Parametric Computation Proofs

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5 Proofs of Claims

5.1 The Proofs of Claims in §2.2

Proposition ($.[.]_{\mathtt{w}}$.1):

(a) $(e_0 \overset{\alpha}{\equiv} e_1 \wedge e_2 \overset{\alpha}{\equiv} e_3) \Rightarrow e_0[x := e_2]_{\mathtt{w}} \overset{\alpha}{\equiv} e_1[x := e_3]_{\mathtt{w}}$

(b) $(\nu_0 \notin \mathrm{FV}(e) \wedge \nu_0 \neq x \neq \nu_1) \Rightarrow e_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}} \overset{\alpha}{\equiv} e_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}$

(c) $e_0[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}} \overset{\alpha}{\equiv} e_0[x := y]_{\mathtt{w}}$ for $\nu \notin \mathrm{FV}(e_0)$

(d) $\lambda z.e_0 \overset{\alpha}{\equiv} \lambda\nu.e_0[z := \nu]_{\mathtt{w}}$ ν fresh

Proof of Proposition ($.[.]_{\mathtt{w}}$.1): The first three properties, (**a**), (**b**) and (**c**) are proved by simultaneous induction on the rank of expression e_0. The next (**d**) is a simple consequence. We establish it first.

(d): To establish (**d**) it suffices (due to rule ($\overset{\alpha}{\equiv}$**.5**)) to show that

$e_0[z := \nu_0]_{\mathtt{w}} \overset{\alpha}{\equiv} e_0[z := \nu]_{\mathtt{w}}[\nu := \nu_0]_{\mathtt{w}}$ for ν and ν_0 fresh.

This is immediate from (**c**).

Base Case: In this case either $e_0 \in \mathbb{A}$ and all three facts are immediate or else $e_0 \in \mathbb{X}$ and we must establish three identities.

(a): The hypothesis that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_0 = e_1$. By computation

$$e_i[x := e_{i+1}]_{\mathtt{w}} = \begin{cases} e_{i+2} & \text{if } e_i = x \\ e_i & \text{otherwise.} \end{cases}$$

Thus the result follows directly from the assumptions.

(b): Assume that $\nu_0 \notin \mathrm{FV}(e)$ and $\nu_1 \neq x$. Then

$$e_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}} = \begin{cases} \nu_1[x := e]_{\mathtt{w}} & \text{if } e_0 = \nu_0 \\ e_0[x := e]_{\mathtt{w}} & \text{if } e_0 \neq \nu_0 \end{cases}$$

$$= \begin{cases} \nu_1 & \text{if } e_0 = \nu_0 \\ e_0[x := e]_{\mathtt{w}} & \text{if } e_0 \neq \nu_0 \end{cases}$$

since $\nu_1 \neq x$

On the other hand

$$e_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} = \begin{cases} e[\nu_0 := \nu_1]_{\mathtt{w}} & \text{if } e_0 = x \\ e_0[\nu_0 := \nu_1]_{\mathtt{w}} & \text{if } e_0 \neq x \end{cases}$$

$$= \begin{cases} e & \text{if } e_0 = x \\ \nu_1 & \text{if } e_0 = \nu_0 \\ e_0 & \text{otherwise} \end{cases}$$

since $\nu_0 \notin \mathrm{FV}(e)$

Clearly these are equal, since $\nu_0 \neq x$.

(c): Assume that $\nu \notin \mathrm{FV}(e_0)$ (i.e $\nu \neq e_0$). Then

$$e_0[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}}$$

$$= \begin{cases} y & \text{if } e_0 = x \\ e_0 & \text{otherwise} \end{cases}$$

$$= e_0[x := y]_{\mathtt{w}}$$

and so we are done.

Induction Step Assume the statement holds for expressions of lesser rank. We consider the (three times) three cases, depending on the construction of e_0.

(1a. $e_0 = \mathtt{app}(e_0', e_0'')$) First observe that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_1 = \mathtt{app}(e_1', e_1'')$ for some e_1' and e_1'', furthermore $e_0' \overset{\alpha}{\equiv} e_1'$ and $e_0'' \overset{\alpha}{\equiv} e_1''$. By computation:

$e_0[x := e_2]_{\mathtt{w}} = \mathtt{app}(e_0', e_0'')[x := e_2]_{\mathtt{w}}$ by assumption

$= \mathtt{app}(e_0'[x := e_2]_{\mathtt{w}}, e_0''[x := e_2]_{\mathtt{w}})$ by definition of $.[.]_{\mathtt{w}}$

$\overset{\alpha}{\equiv} \mathtt{app}(e_1'[x := e_3]_{\mathtt{w}}, e_1''[x := e_3]_{\mathtt{w}})$ by the above, the induction hypothesis and the rule ($\overset{\alpha}{\equiv}$**.2**)

$= \mathtt{app}(e_1', e_1'')[x := e_3]_{\mathtt{w}}$ by definition of $.[.]_{\mathtt{w}}$

$= e_1[x := e_3]_{\mathtt{w}}$ by assumption

(1b. $e_0 = \mathtt{app}(e_0', e_0'')$) Assume that $\nu_0 \notin \mathrm{FV}(e)$ and $\nu_1 \neq x$.

$e_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}} = \mathtt{app}(e_0', e_0'')[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}$ by assumption

$= \mathtt{app}(e_0'[\nu_0 := \nu_1]_{\mathtt{w}}, e_0''[\nu_0 := \nu_1]_{\mathtt{w}})[x := e]_{\mathtt{w}}$ by definition of $.[.]_{\mathtt{w}}$

$= \mathtt{app}(e_0'[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}, e_0''[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}})$ again by definition of $.[.]_{\mathtt{w}}$

$\overset{\alpha}{\equiv} \mathtt{app}(e_0'[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}, e_0''[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})$ by the induction hypothesis and rule ($\overset{\alpha}{\equiv}$**.2**)

$= \mathtt{app}(e_0'[x := e]_{\mathtt{w}}, e_0''[x := e]_{\mathtt{w}})[\nu_0 := \nu_1]_{\mathtt{w}}$ again by definition of $.[.]_{\mathtt{w}}$

$= \mathtt{app}(e_0', e_0'')[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}$ yet again by definition of $.[.]_{\mathtt{w}}$

$= e_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}$ by assumption

(1c. $e_0 = \mathtt{app}(e_0', e_0'')$**)** Assume $\nu \notin \mathrm{FV}(e_0)$. Then

$$
\begin{aligned}
e_0[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}} &= \mathtt{app}(e_0', e_0'')[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}} && \text{by assumption}\\
&= \mathtt{app}(e_0'[\nu := y]_{\mathtt{w}}, e_0''[\nu := y]_{\mathtt{w}})[x := \nu]_{\mathtt{w}} && \text{by definition of } .[.]_{\mathtt{w}}\\
&= \mathtt{app}(e_0'[\nu := y]_{\mathtt{w}}[x := \nu]_{\mathtt{w}}, e_0''[\nu := y]_{\mathtt{w}}[x := \nu]_{\mathtt{w}}) && \text{again by definition of } .[.]_{\mathtt{w}}\\
&\overset{\alpha}{\equiv} \mathtt{app}(e_0'[x := y]_{\mathtt{w}}, e_0''[x := y]_{\mathtt{w}}) && \text{by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.2})\\
&= \mathtt{app}(e_0', e_0'')[x := y]_{\mathtt{w}} && \text{again by definition of } .[.]_{\mathtt{w}}\\
&= e_0[x := y]_{\mathtt{w}} && \text{by assumption}
\end{aligned}
$$

(2a. $e_0 = \bullet^{\sigma_0'}$**)** Again the hypothesis that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_1 = \bullet^{\sigma_1'}$ for some σ_1' and $\sigma_0' \overset{\alpha}{\equiv} \sigma_1'$. Now by computation:

$$e_i[x := e_{i+2}]_{\mathtt{w}} = \bullet^{\sigma_i'}[x := e_{i+2}]_{\mathtt{w}} = \bullet^{\sigma_i'[x := e_{i+2}]_{\mathtt{w}}}$$

So it suffices by rule ($\overset{\alpha}{\equiv}$**.4**) to show that $\sigma_0'[x := e_2]_{\mathtt{w}} \overset{\alpha}{\equiv} \sigma_1'[x := e_3]_{\mathtt{w}}$. The assumption that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $\mathrm{Dom}(\sigma_0') = \mathrm{Dom}(\sigma_1')$ and thus using rules ($\overset{\alpha}{\equiv}$**.3**) & ($\overset{\alpha}{\equiv}$**.4**) the problem reduces to showing that

$$(\forall z \in \mathrm{Dom}(\sigma_i'))(\sigma_0'(z)[x := e_2]_{\mathtt{w}} \overset{\alpha}{\equiv} \sigma_1'(z)[x := e_3]_{\mathtt{w}})$$

which follows from the induction hypothesis.

(2b. $e_0 = \bullet^{\sigma_0}$**)** Assume that $\nu_0 \notin \mathrm{FV}(e)$ and $\nu_1 \neq x$, and observe that

$$e_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}} = \bullet^{\sigma_0}[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}} = \bullet^{\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}}[x := e]_{\mathtt{w}} = \bullet^{\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}}$$
$$e_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} = \bullet^{\sigma_0}[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} = \bullet^{\sigma_0[x := e]_{\mathtt{w}}}[\nu_0 := \nu_1]_{\mathtt{w}} = \bullet^{\sigma_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}}$$

So it suffices by rule ($\overset{\alpha}{\equiv}$**.4**) to show

$$\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}} \overset{\alpha}{\equiv} \sigma_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}$$

Now since

$$\mathrm{Dom}(\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}) = \mathrm{Dom}(\sigma_0) = \mathrm{Dom}(\sigma_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})$$

it suffices by rule ($\overset{\alpha}{\equiv}$**.3**) to show that for any $z \in \mathrm{Dom}(\sigma_0)$ that

$$\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}(z) \overset{\alpha}{\equiv} \sigma_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}(z)$$

and this is a simple application of the induction hypothesis.

(2c. $e_0 = \bullet^{\sigma_0}$**)** Assume $\nu \notin \mathrm{FV}(e_0)$. Then note that

$$e_0[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}} = \bullet^{\sigma_0}[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}} = \bullet^{\sigma_0[x := \nu]_{\mathtt{w}}}[\nu := y]_{\mathtt{w}} = \bullet^{\sigma_0[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}}}$$
$$e_0[x := y]_{\mathtt{w}} = \bullet^{\sigma_0}[x := y]_{\mathtt{w}} = \bullet^{\sigma_0[x := y]_{\mathtt{w}}}$$

So again using rules ($\overset{\alpha}{\equiv}$**.4**) and ($\overset{\alpha}{\equiv}$**.3**) it suffices to show that for any $x \in \mathrm{Dom}(\sigma_0)$ that

$$\sigma_0[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}}(x) \overset{\alpha}{\equiv} \sigma_0[x := y]_{\mathtt{w}}(x)$$

again a simple application of the induction hypothesis, noting that $\nu \notin \mathrm{FV}(\sigma_0(x)) \subseteq \mathrm{FV}(e_0)$.

(3a. $e_0 = \lambda x_0.e'_0$**)** First observe the assumption that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_1 = \lambda x_1.e'_1$ for suitable x_1 and e'_1, furthermore $e'_0[x_0 := \nu]_{\mathtt{w}} \overset{\alpha}{\equiv} e'_1[x_1 := \nu]_{\mathtt{w}}$ for ν fresh. Now for $i < 2$ let $\nu_i = \mathbf{fresh}(\{e_i, \{x := e_{i+2}\}\})$. Then $e_i[x := e_{i+2}]_{\mathtt{w}} = \lambda\nu_i.(e'_i[x_i := \nu_i]_{\mathtt{w}}[x := e_{i+2}]_{\mathtt{w}})$ Thus to demonstrate that $e_0[x := e_2]_{\mathtt{w}} \overset{\alpha}{\equiv} e_1[x := e_3]_{\mathtt{w}}$ it suffices (by an application of rule ($\overset{\alpha}{\equiv}$**.5**)) to demonstrate that

$$e'_0[x_0 := \nu_0]_{\mathtt{w}}[x := e_2]_{\mathtt{w}}[\nu_0 := \nu_3]_{\mathtt{w}} \overset{\alpha}{\equiv} e'_1[x_1 := \nu_1]_{\mathtt{w}}[x := e_3]_{\mathtt{w}}[\nu_1 := \nu_3]_{\mathtt{w}}$$

for ν_3 fresh. Now by computation:

$e'_0[x_0 := \nu_0]_{\mathtt{w}}[x := e_2]_{\mathtt{w}}[\nu_0 := \nu_3]_{\mathtt{w}}$

$\overset{\alpha}{\equiv} e'_0[x_0 := \nu_0]_{\mathtt{w}}[\nu_0 := \nu_3]_{\mathtt{w}}[x := e_2]_{\mathtt{w}}$ by (**b**) of the induction hypothesis

$\overset{\alpha}{\equiv} e'_0[x_0 := \nu_3]_{\mathtt{w}}[x := e_2]_{\mathtt{w}}$ by (**a**) and (**c**) of the induction hypothesis

$\overset{\alpha}{\equiv} e'_1[x_1 := \nu_3]_{\mathtt{w}}[x := e_3]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and the initial observations

$\overset{\alpha}{\equiv} e'_1[x_1 := \nu_1]_{\mathtt{w}}[\nu_1 := \nu_3]_{\mathtt{w}}[x := e_3]_{\mathtt{w}}$ by (**a**) and (**c**) of the induction hypothesis

$\overset{\alpha}{\equiv} e'_1[x_1 := \nu_1]_{\mathtt{w}}[x := e_3]_{\mathtt{w}}[\nu_1 := \nu_3]_{\mathtt{w}}$ by (**b**) of the induction hypothesis

(3b. $e_0 = \lambda z.e'_0$**)** Assume that $\nu_0 \notin \mathrm{FV}(e)$ and $\nu_1 \neq x$. Let

$$\nu_2 = \mathbf{fresh}(\{e_0, \{\nu_0 := \nu_1\}\})$$
$$\nu_3 = \mathbf{fresh}(\{\lambda\nu_2.(e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}), \{x := e\}\})$$
$$\nu_4 = \mathbf{fresh}(\{e_0, \{x := e\}\})$$
$$\nu_5 = \mathbf{fresh}(\{\lambda\nu_4.(e'_0[z := \nu_4]_{\mathtt{w}}[x := e]_{\mathtt{w}}), \{\nu_0 := \nu_1\}\})$$

Then by computation:

$$e_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}} = \lambda z.e'_0[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}$$
$$= (\lambda\nu_2.e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})[x := e]_{\mathtt{w}}$$
$$= \lambda\nu_3.(e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[x := e]_{\mathtt{w}})$$

While on the other hand

$$e_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} = \lambda z.e'_0[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}$$
$$= (\lambda\nu_4.e'_0[z := \nu_4]_{\mathtt{w}}[x := e]_{\mathtt{w}})[\nu_0 := \nu_1]_{\mathtt{w}}$$
$$= \lambda\nu_5.(e'_0[z := \nu_4]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})$$

Consequently by choosing ν fresh with respect to

$$\lambda\nu_3.(e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[x := e]_{\mathtt{w}}) \quad \text{and} \quad \lambda\nu_5.(e'_0[z := \nu_4]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})$$

we are reduced (by rule ($\overset{\alpha}{\equiv}$**.5**)) to showing the α-equivalence of:

$$e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_3 := \nu]_{\mathtt{w}}$$

and

$$e'_0[z := \nu_4]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}$$

Now by computation:

$$e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_3 := \nu]_{\mathtt{w}} \stackrel{\alpha}{\equiv} e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[\nu_3 := \nu]_{\mathtt{w}}[x := e]_{\mathtt{w}}$$

by (**b**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu]_{\mathtt{w}}[x := e]_{\mathtt{w}}$ by (**a**) and (**c**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e'_0[z := \nu_2]_{\mathtt{w}}[\nu_2 := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}$ by (**a**) and (**b**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e'_0[z := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}$ by (**a**) and (**c**) of the induction hypothesis

On the other hand

$$e'_0[z := \nu_4]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}} \stackrel{\alpha}{\equiv} e'_0[z := \nu_4]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}$$

by (**a**) and (**b**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e'_0[z := \nu_5]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}$ by (**a**) and (**c**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e'_0[z := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}$ by (**a**) and (**b**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e'_0[z := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}[x := e]_{\mathtt{w}}$ by (**b**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e'_0[z := \nu_5]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}$ by (**a**) and (**b**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e'_0[z := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[x := e]_{\mathtt{w}}$ by (**a**) and (**c**) of the induction hypothesis

(3c. $e_0 = \lambda z.e'_0$**)** Assume $\nu \notin \mathrm{FV}(e_0)$. Then

$e_0[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}} = (\lambda z.e'_0)[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}}$ by assumption

$= \lambda\nu_0.(e'_0[z := \nu_0]_{\mathtt{w}}[x := \nu]_{\mathtt{w}})[\nu := y]_{\mathtt{w}}$ for ν_0 fresh, by definition of $.[.]_{\mathtt{w}}$

$= \lambda\nu_1.(e'_0[z := \nu_0]_{\mathtt{w}}[x := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu := y]_{\mathtt{w}})$ for ν_1 fresh, by definition of $.[.]_{\mathtt{w}}$

$\stackrel{\alpha}{\equiv} \lambda\nu_1.(e'_0[z := \nu_0]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}})$ by the induction hypothesis w.r.t. (**b**) and (**a**)

$\stackrel{\alpha}{\equiv} \lambda\nu_1.(e'_0[z := \nu_1]_{\mathtt{w}}[x := \nu]_{\mathtt{w}}[\nu := y]_{\mathtt{w}})$ by the induction hypothesis w.r.t. (**c**) and (**a**)

$\stackrel{\alpha}{\equiv} \lambda\nu_1.(e'_0[z := \nu_1]_{\mathtt{w}}[x := y]_{\mathtt{w}})$ again by the induction hypothesis w.r.t. (**c**) and (**a**)

$= (\lambda z.e'_0)[x := y]_{\mathtt{w}})$ by definition of $.[.]_{\mathtt{w}}$

$= e_0[x := y]_{\mathtt{w}}$ by assumption

□

Proposition ($\stackrel{\alpha}{\equiv}$**.1):**

(a) $(e_0 \stackrel{\alpha}{\equiv} e_1 \wedge \sigma_0 \stackrel{\alpha}{\equiv} \sigma_1) \Rightarrow e_0[\sigma_0]_{\mathtt{w}} \stackrel{\alpha}{\equiv} e_1[\sigma_1]_{\mathtt{w}}$

(b) $(\nu_0 \notin \mathrm{V}(\sigma) \wedge \nu_1 \notin \mathrm{Dom}(\sigma)) \Rightarrow e_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} \stackrel{\alpha}{\equiv} e_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}$

(c) $(\lambda\nu.e_0)[\sigma]_{\mathtt{w}} \stackrel{\alpha}{\equiv} \lambda\nu.(e_0[\sigma]_{\mathtt{w}})$ if $\nu \notin \mathrm{V}(\sigma)$

Proof of Proposition ($\stackrel{\alpha}{\equiv}$**.1):** The first two properties, (**a**) and (**b**) are proved by simultaneous induction on the rank of expression e_0. The next (**c**) is a simple consequence. We establish it first.

(c): Suppose that $\nu \notin \mathrm{V}(\sigma)$ then

$(\lambda\nu.e_0)[\sigma]_{\mathtt{w}} = \lambda\nu'.(e_0[\nu := \nu']_{\mathtt{w}}[\sigma]_{\mathtt{w}})$

by definition of $.[.]_{\mathtt{w}}$ for $\nu' = \mathbf{fresh}(\{\lambda\nu.e_0, \sigma\})$. So using rule ($\overset{\alpha}{\equiv}$**.5**) it suffices to show that:

$$e_0[\nu := \nu']_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu' := \nu'']_{\mathtt{w}} \overset{\alpha}{\equiv} e_0[\sigma]_{\mathtt{w}}[\nu := \nu'']_{\mathtt{w}} \qquad \text{for } \nu'' \text{ fresh}$$

Now to this end we have:

$$e_0[\nu := \nu']_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu' := \nu'']_{\mathtt{w}} \overset{\alpha}{\equiv} e_0[\nu := \nu']_{\mathtt{w}}[\nu' := \nu'']_{\mathtt{w}}[\sigma]_{\mathtt{w}}$$

by (**b**) since ν' and ν'' are both fresh, in particular $\nu' \notin \mathrm{V}(\sigma)$ and $\nu'' \notin \mathrm{Dom}(\sigma)$

$$\overset{\alpha}{\equiv} e_0[\nu := \nu'']_{\mathtt{w}}[\sigma]_{\mathtt{w}}$$

by (**a**) and (**c**) of proposition ($.[.]_{\mathtt{w}}$**.1**) since $\nu'' \notin \mathrm{FV}(e_0)$

$$\overset{\alpha}{\equiv} e_0[\sigma]_{\mathtt{w}}[\nu := \nu'']_{\mathtt{w}}$$

by (**b**) since $\nu'' \notin \mathrm{Dom}(\sigma)$ and by assumption $\nu \notin \mathrm{FV}(\sigma)$

Base Case: In this case either $e_0 \in \mathbb{A}$ and the results are immediate or else $e_0 \in \mathbb{X}$ and we must establish two identities.

(a): The hypothesis that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_0 = e_1$, and the hypothesis that $\sigma_0 \overset{\alpha}{\equiv} \sigma_1$ implies that $\mathrm{Dom}(\sigma_0) = \mathrm{Dom}(\sigma_1)$ and that for each $x \in \mathrm{Dom}(\sigma_0)$ we have that $\sigma_0(x) \overset{\alpha}{\equiv} \sigma_1(x)$. By computation

$$e_i[\sigma_i]_{\mathtt{w}} = \begin{cases} \sigma_i(e_i) & \text{if } e_i \in \mathrm{Dom}(\sigma_i) \\ e_i & \text{otherwise.} \end{cases}$$

Thus the result in this case reduces to showing that $\sigma_0(e_0) \overset{\alpha}{\equiv} \sigma_1(e_1)$ when $e_i \in \mathrm{Dom}(\sigma_i)$, and this follows directly from the above mentioned facts.

(b): Assume that $\nu_0 \notin \mathrm{V}(\sigma)$ and $\nu_1 \notin \mathrm{Dom}(\sigma)$. Then

$$e_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} = \begin{cases} \nu_1[\sigma]_{\mathtt{w}} & \text{if } e_0 = \nu_0 \\ e_0[\sigma]_{\mathtt{w}} & \text{if } e_0 \neq \nu_0 \end{cases}$$

$$= \begin{cases} \nu_1 & \text{if } e_0 = \nu_0 \\ e_0[\sigma]_{\mathtt{w}} & \text{if } e_0 \neq \nu_0 \end{cases}$$

since $\nu_1 \notin \mathrm{Dom}(\sigma)$

On the other hand

$$e_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} = \begin{cases} \sigma(e_0)[\nu_0 := \nu_1]_{\mathtt{w}} & \text{if } e_0 \in \mathrm{Dom}(\sigma) \\ e_0[\nu_0 := \nu_1]_{\mathtt{w}} & \text{if } e_0 \notin \mathrm{Dom}(\sigma) \end{cases}$$

$$= \begin{cases} \sigma(e_0) & \text{if } e_0 \in \mathrm{Dom}(\sigma) \\ \nu_1 & \text{if } e_0 = \nu_0 \\ e_0 & \text{otherwise} \end{cases}$$

since $\nu_0 \notin \mathrm{V}(\sigma)$

Clearly these are equal.

Induction Step Assume the statement holds for expressions of lesser rank. We consider the (two times) three cases, depending on the construction of e_0.

(1a. $e_0 = \mathtt{app}(e_0', e_0'')$**)** First observe that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_1 = \mathtt{app}(e_1', e_1'')$ for some e_1' and e_1'', furthermore $e_0' \overset{\alpha}{\equiv} e_1'$ and $e_0'' \overset{\alpha}{\equiv} e_1''$. By computation:

$$
\begin{aligned}
e_0[\sigma_0]_{\mathtt{w}} &= \mathtt{app}(e_0', e_0'')[\sigma_0]_{\mathtt{w}} && \text{by assumption} \\
&= \mathtt{app}(e_0'[\sigma_0]_{\mathtt{w}}, e_0''[\sigma_0]_{\mathtt{w}}) && \text{by definition of } .[.]_{\mathtt{w}} \\
&\overset{\alpha}{\equiv} \mathtt{app}(e_1'[\sigma_1]_{\mathtt{w}}, e_1''[\sigma_1]_{\mathtt{w}}) && \text{by the above, the induction hypothesis and the rule } (\overset{\alpha}{\equiv}\textbf{.2}) \\
&= \mathtt{app}(e_1', e_1'')[\sigma_1]_{\mathtt{w}} && \text{by definition of } .[.]_{\mathtt{w}} \\
&= e_0[\sigma_0]_{\mathtt{w}} && \text{by assumption}
\end{aligned}
$$

(1b. $e_0 = \mathtt{app}(e_0', e_0'')$**)** Assume that $\nu_0 \notin \mathrm{V}(\sigma)$ and $\nu_1 \notin \mathrm{Dom}(\sigma)$.

$$
\begin{aligned}
e_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} &= \mathtt{app}(e_0', e_0'')[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} && \text{by assumption} \\
&= \mathtt{app}(e_0'[\nu_0 := \nu_1]_{\mathtt{w}}, e_0''[\nu_0 := \nu_1]_{\mathtt{w}})[\sigma]_{\mathtt{w}} && \text{by definition of } .[.]_{\mathtt{w}} \\
&= \mathtt{app}(e_0'[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}, e_0''[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}) && \text{again by definition of } .[.]_{\mathtt{w}} \\
&\overset{\alpha}{\equiv} \mathtt{app}(e_0'[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}, e_0''[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}) && \text{by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.2}) \\
&= \mathtt{app}(e_0'[\sigma]_{\mathtt{w}}, e_0''[\sigma]_{\mathtt{w}})[\nu_0 := \nu_1]_{\mathtt{w}} && \text{again by definition of } .[.]_{\mathtt{w}} \\
&= \mathtt{app}(e_0', e_0'')[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} && \text{yet again by definition of } .[.]_{\mathtt{w}} \\
&= e_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} && \text{by assumption}
\end{aligned}
$$

(2a. $e_0 = \bullet^{\sigma_0'}$**)** Again the hypothesis that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_1 = \bullet^{\sigma_1'}$ for some σ_1' and $\sigma_0' \overset{\alpha}{\equiv} \sigma_1'$. Now by computation:

$$e_i[\sigma_i]_{\mathtt{w}} = \bullet^{\sigma_i'}[\sigma_i]_{\mathtt{w}} = \bullet^{\sigma_i'[\sigma_i]_{\mathtt{w}}}$$

So it suffices by rule ($\overset{\alpha}{\equiv}$**.4**) to show that $\sigma_0'[\sigma_0]_{\mathtt{w}} \overset{\alpha}{\equiv} \sigma_1'[\sigma_1]_{\mathtt{w}}$. The assumption that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $\mathrm{Dom}(\sigma_0') = \mathrm{Dom}(\sigma_1')$ and thus using rules ($\overset{\alpha}{\equiv}$**.3**) & ($\overset{\alpha}{\equiv}$**.4**) the problem reduces to showing that

$$(\forall x \in \mathrm{Dom}(\sigma_i'))(\sigma_0'(x)[\sigma_0]_{\mathtt{w}} \overset{\alpha}{\equiv} \sigma_1'(x)[\sigma_1]_{\mathtt{w}})$$

which follows from the induction hypothesis.

(2b. $e_0 = \bullet^{\sigma_0}$**)** Assume that $\nu_0 \notin \mathrm{V}(\sigma)$ and $\nu_1 \notin \mathrm{Dom}(\sigma)$, and observe that

$$
\begin{aligned}
e_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} &= \bullet^{\sigma_0}[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} = \bullet^{\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}}[\sigma]_{\mathtt{w}} = \bullet^{\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}} \\
e_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} &= \bullet^{\sigma_0}[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} = \bullet^{\sigma_0[\sigma]_{\mathtt{w}}}[\nu_0 := \nu_1]_{\mathtt{w}} = \bullet^{\sigma_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}}
\end{aligned}
$$

So it suffices by rule ($\overset{\alpha}{\equiv}$**.4**) to show

$$\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} \overset{\alpha}{\equiv} \sigma_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}$$

Now since

$$\mathrm{Dom}(\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}) = \mathrm{Dom}(\sigma_0) = \mathrm{Dom}(\sigma_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})$$

it suffices by rule ($\overset{\alpha}{\equiv}$**.3**) to show that for any $x \in \mathrm{Dom}(\sigma_0)$ that

$$\sigma_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}(x) \overset{\alpha}{\equiv} \sigma_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}(x)$$

and this is a simple application of the induction hypothesis.

(3a. $e_0 = \lambda x_0.e'_0$**)** First observe the assumption that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_1 = \lambda x_1.e'_1$ for suitable x_1 and e'_1, furthermore $e'_0[x_0 := \nu]_{\mathtt{w}} \overset{\alpha}{\equiv} e'_1[x_1 := \nu]_{\mathtt{w}}$ for ν fresh. Also note by lemma ($\overset{\alpha}{\equiv}$**.0**) that $e'_i[x_i := \nu]_{\mathtt{w}}$ is of lesser rank than e_i. Now for $i < 2$ let $\nu_i = \mathbf{fresh}(\{e_i, \sigma_i\})$. Then $e_i[\sigma_i]_{\mathtt{w}} = \lambda\nu_i.(e'_i[x_i := \nu_i]_{\mathtt{w}}[\sigma_i]_{\mathtt{w}})$ Thus to demonstrate that $e_0[\sigma_0]_{\mathtt{w}} \overset{\alpha}{\equiv} e_1[\sigma_1]_{\mathtt{w}}$ it suffices (by an application of rule ($\overset{\alpha}{\equiv}$**.5**)) to demonstrate that

$$e'_0[x_0 := \nu_0]_{\mathtt{w}}[\sigma_0]_{\mathtt{w}}[\nu_0 := \nu_3]_{\mathtt{w}} \overset{\alpha}{\equiv} e'_1[x_1 := \nu_1]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}[\nu_1 := \nu_3]_{\mathtt{w}}$$

for ν_3 fresh. Now by computation:

$e'_0[x_0 := \nu_0]_{\mathtt{w}}[\sigma_0]_{\mathtt{w}}[\nu_0 := \nu_3]_{\mathtt{w}}$

$\overset{\alpha}{\equiv} e'_0[x_0 := \nu_0]_{\mathtt{w}}[\nu_0 := \nu_3]_{\mathtt{w}}[\sigma_0]_{\mathtt{w}}$ by (**b**) of the induction hypothesis

$\overset{\alpha}{\equiv} e'_0[x_0 := \nu_3]_{\mathtt{w}}[\sigma_0]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and (**c**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

$\overset{\alpha}{\equiv} e'_1[x_1 := \nu_3]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and the initial observations

$\overset{\alpha}{\equiv} e'_1[x_1 := \nu_1]_{\mathtt{w}}[\nu_1 := \nu_3]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and (**c**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

$\overset{\alpha}{\equiv} e'_1[x_1 := \nu_1]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}[\nu_1 := \nu_3]_{\mathtt{w}}$ by (**b**) of the induction hypothesis

(3b. $e_0 = \lambda z.e'_0$**)** Assume that $\nu_0 \notin \mathrm{V}(\sigma)$ and $\nu_1 \notin \mathrm{Dom}(\sigma)$. Let

$$\nu_2 = \mathbf{fresh}(\{e_0, \{\nu_0 := \nu_1\}\})$$
$$\nu_3 = \mathbf{fresh}(\{\lambda\nu_2.(e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}), \sigma\})$$
$$\nu_4 = \mathbf{fresh}(\{e_0, \sigma\})$$
$$\nu_5 = \mathbf{fresh}(\{\lambda\nu_4.(e'_0[z := \nu_4]_{\mathtt{w}}[\sigma]_{\mathtt{w}}), \{\nu_0 := \nu_1\}\})$$

Then by computation:

$$e_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} = \lambda z.e'_0[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}$$
$$= (\lambda\nu_2.e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})[\sigma]_{\mathtt{w}}$$
$$= \lambda\nu_3.(e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[\sigma]_{\mathtt{w}})$$

While on the other hand

$$e_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} = \lambda z.e'_0[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}$$
$$= (\lambda\nu_4.e'_0[z := \nu_4]_{\mathtt{w}}[\sigma]_{\mathtt{w}})[\nu_0 := \nu_1]_{\mathtt{w}}$$
$$= \lambda\nu_5.(e'_0[z := \nu_4]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})$$

Consequently by choosing ν fresh with respect to

$$\lambda\nu_3.(e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[\sigma]_{\mathtt{w}}) \quad \text{and} \quad \lambda\nu_5.(e'_0[z := \nu_4]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}})$$

we are reduced (by rule ($\overset{\alpha}{\equiv}$**.5**)) to showing the α-equivalence of:

$$e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_3 := \nu]_{\mathtt{w}} \quad \text{and} \quad e'_0[z := \nu_4]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}$$

Now by computation:

$$e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_3 := \nu]_{\mathtt{w}} \overset{\alpha}{\equiv} e'_0[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu_3]_{\mathtt{w}}[\nu_3 := \nu]_{\mathtt{w}}[\sigma]_{\mathtt{w}}$$

by (**b**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e_0'[z := \nu_2]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_2 := \nu]_{\mathtt{w}}[\sigma]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and (**c**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

$\stackrel{\alpha}{\equiv} e_0'[z := \nu_2]_{\mathtt{w}}[\nu_2 := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and (**b**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

$\stackrel{\alpha}{\equiv} e_0'[z := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and (**c**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

On the other hand

$e_0'[z := \nu_4]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}} \stackrel{\alpha}{\equiv} e_0'[z := \nu_4]_{\mathtt{w}}[\nu_4 := \nu_5]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}$

by (**b**) of the induction hypothesis and two applications of (**a**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

$\stackrel{\alpha}{\equiv} e_0'[z := \nu_5]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and (**c**) and (**a**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

$\stackrel{\alpha}{\equiv} e_0'[z := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}$ by (**b**) of the induction hypothesis and (**a**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

$\stackrel{\alpha}{\equiv} e_0'[z := \nu_5]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}[\sigma]_{\mathtt{w}}$ by (**b**) of the induction hypothesis

$\stackrel{\alpha}{\equiv} e_0'[z := \nu_5]_{\mathtt{w}}[\nu_5 := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and (**b**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

$\stackrel{\alpha}{\equiv} e_0'[z := \nu]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}}$ by (**a**) of the induction hypothesis and (**c**) of proposition ($.[.]_{\mathtt{w}}$**.1**)

□

Proof of Proposition ($.[.]_{\mathtt{w}}$.2): The proof is a simple induction on $\rho(e)$.

Base Case: In this case either $e \in \mathbb{A}$ and the result is trivial or else $e \in \mathbb{X}$ and consequently

$$e[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}} = \begin{cases} \sigma_0(e)[\sigma_1]_{\mathtt{w}} & \text{if } e \in \mathrm{Dom}(\sigma_1) \\ e & \text{otherwise} \end{cases} \qquad \text{since } (\mathrm{Dom}(\sigma_1) - \mathrm{Dom}(\sigma_0)) \cap \mathrm{FV}(e) = \emptyset$$
$$= e[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}$$

Induction Step Assume the statement holds for expressions of lesser rank. We consider the three cases, depending on the construction of e.

(1. $e = \mathtt{app}(e_0, e_1)$)

$e[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}} = \mathtt{app}(e_0, e_1)[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}$ by assumption

$= \mathtt{app}(e_0[\sigma_0]_{\mathtt{w}}, e_1[\sigma_0]_{\mathtt{w}})[\sigma_1]_{\mathtt{w}}$ by definition of $.[.]_{\mathtt{w}}$

$= \mathtt{app}(e_0[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}, e_1[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}})$ again by definition of $.[.]_{\mathtt{w}}$

$\stackrel{\alpha}{\equiv} \mathtt{app}(e_0[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}, e_1[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}})$ by the induction hypothesis and rule ($\stackrel{\alpha}{\equiv}$**.2**)

$= \mathtt{app}(e_0, e_1)[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}$ by definition of $.[.]_{\mathtt{w}}$

$= e[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}$ by assumption

(2. $e = \bullet^{\sigma}$) In this case

$e[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}} = \bullet^{\sigma}[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}} = \bullet^{\sigma[\sigma_0]_{\mathtt{w}}}[\sigma_1]_{\mathtt{w}} = \bullet^{\sigma[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}}$

$e[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}} = \bullet^{\sigma}[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}} = \bullet^{\sigma[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}}$

Thus by rule ($\stackrel{\alpha}{\equiv}$**.4**) it suffices to show that

$\sigma[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}} \stackrel{\alpha}{\equiv} \sigma[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}$

First observe that

$$\mathrm{Dom}(\sigma[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}) = \mathrm{Dom}(\sigma) = \mathrm{Dom}(\sigma[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}})$$

So it suffices by rule ($\stackrel{\alpha}{\equiv}$**.3**) to show that for every $x \in \mathrm{Dom}(\sigma)$ that

$$\sigma[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}(x) \stackrel{\alpha}{\equiv} \sigma[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}(x)$$

Now

$$\begin{aligned}
\sigma[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}(x) &= \sigma(x)[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}} && \text{by definition of } .[.]_{\mathtt{w}} \\
&\stackrel{\alpha}{\equiv} \sigma(x)[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}} && \text{by the induction hypothesis, noting that } \mathrm{FV}(\sigma(x)) \subseteq \mathrm{FV}(e) \\
&= \sigma[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}(x)
\end{aligned}$$

(3. $e = \lambda x.e_0$**)**

$$\begin{aligned}
e[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}} &= (\lambda x.e_0)[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}} \\
&= \lambda\nu.(e_0[x := \nu]_{\mathtt{w}}[\sigma_0]_{\mathtt{w}})[\sigma_1]_{\mathtt{w}} && \text{by definition of } .[.]_{\mathtt{w}} \text{ for } \nu \text{ fresh} \\
&= \lambda\nu'.(e_0[x := \nu]_{\mathtt{w}}[\sigma_0]_{\mathtt{w}}[\nu := \nu']_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}) && \text{again by definition of } .[.]_{\mathtt{w}} \text{ for } \nu' \text{ fresh} \\
&\stackrel{\alpha}{\equiv} \lambda\nu'.(e_0[x := \nu]_{\mathtt{w}}[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}[\nu := \nu']_{\mathtt{w}}) && \text{by (**a**) of proposition } (.[.]_{\mathtt{w}}\textbf{.1}) \text{ and rule } (\stackrel{\alpha}{\equiv}\textbf{.6}) \\
&\stackrel{\alpha}{\equiv} \lambda\nu.(e_0[x := \nu]_{\mathtt{w}}[\sigma_0]_{\mathtt{w}}[\sigma_1]_{\mathtt{w}}) && \text{by (**c**) of proposition } (.[.]_{\mathtt{w}}\textbf{.1}) \\
&\stackrel{\alpha}{\equiv} \lambda\nu.(e_0[x := \nu]_{\mathtt{w}}[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}}) && \text{by the induction hypothesis and rule } (\stackrel{\alpha}{\equiv}\textbf{.6}) \\
&= (\lambda x.e_0)[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}} && \text{by definition of } .[.]_{\mathtt{w}} \\
&= e[\sigma_0[\sigma_1]_{\mathtt{w}}]_{\mathtt{w}} && \text{by assumption}
\end{aligned}$$

□

5.2 The Proofs of Claims in §2.3

Proposition ($\stackrel{\alpha}{\equiv}$.2):

$$e_0 \stackrel{\alpha}{\equiv} e_1 \;\Rightarrow\; \mathbf{Trap}(e_0, X) \stackrel{\alpha}{\equiv} \mathbf{Trap}(e_1, X)$$

Proof of Proposition ($\stackrel{\alpha}{\equiv}$.2): The proof is a simple induction on $\rho(e_0) = \rho(e_1)$.

Base Case: In this case $e_i \in \mathbb{A} \cup \mathbb{X}$ consequently the assumption that $e_0 \stackrel{\alpha}{\equiv} e_1$ implies that $e_0 = e_1$. By computation:

$$\mathbf{Trap}(e_0, X) = e_0 = e_1 = \mathbf{Trap}(e_1, X)$$

and hence the desired conclusion.

Induction Step Assume the statement holds for expressions of lesser rank. We consider the three cases, depending on the construction of e_i.

(1. $e_i = \mathtt{app}(e_i', e_i'')$) First observe that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $e_0' \overset{\alpha}{\equiv} e_1'$ and $e_0'' \overset{\alpha}{\equiv} e_1''$. By computation:

$\mathbf{Trap}(e_0, X) = \mathbf{Trap}(\mathtt{app}(e_0', e_0''), X)$ by assumption

$= \mathtt{app}(\mathbf{Trap}(e_0', X), \mathbf{Trap}(e_0'', X))$ by definition of $\mathbf{Trap}$

$\overset{\alpha}{\equiv} \mathtt{app}(\mathbf{Trap}(e_1', X), \mathbf{Trap}(e_1'', X))$ by the induction hypothesis and rule ($\overset{\alpha}{\equiv}$**.2**)

$= \mathbf{Trap}(\mathtt{app}(e_1', e_1''), X)$ by definition of $\mathbf{Trap}$

$= \mathbf{Trap}(e_0, X)$ by assumption

(2. $e_i = \bullet^{\sigma_i}$) By computation:

$\mathbf{Trap}(e_0, X) = \mathbf{Trap}(\bullet^{\sigma_0}, X)$ by assumption

$= \bullet^{\mathbf{Trap}(\sigma_0, X)}$ by definition of $\mathbf{Trap}$

$\overset{\alpha}{\equiv} \bullet^{\mathbf{Trap}(\sigma_1, X)}$ by induction hypothesis and rules ($\overset{\alpha}{\equiv}$**.3**) & ($\overset{\alpha}{\equiv}$**.4**)

$= \mathbf{Trap}(\bullet^{\sigma_1}, X)$ by definition of $\mathbf{Trap}$

$= \mathbf{Trap}(e_1, X)$ by assumption

(3. $e_i = \lambda x_i.e_i'$) As usual observe the assumption that $e_0 \overset{\alpha}{\equiv} e_1$ implies $e_0'[x_0 := \nu]_{\mathtt{w}} \overset{\alpha}{\equiv} e_1'[x_1 := \nu]_{\mathtt{w}}$ for ν fresh. Since either $e_0 \overset{\alpha}{\equiv} e_1$ is established by rule ($\overset{\alpha}{\equiv}$**.5**) or by ($\overset{\alpha}{\equiv}$**.6**). In the former case the result is immediate, while in the later it follows from proposition ($\overset{\alpha}{\equiv}$**.1**). Also note that $e_i'[x_i := \nu]_{\mathtt{w}}$ is of lesser rank than e_i. By computation:

$\mathbf{Trap}(e_0, X) = \mathbf{Trap}(\lambda x_0.e_0', X)$ by assumption

$= \lambda\nu.\mathbf{Trap}(e_0'[x_0 := \nu]_{\mathtt{w}}, X)$ by definition of $\mathbf{Trap}$, for ν fresh

$\overset{\alpha}{\equiv} \lambda\nu.\mathbf{Trap}(e_1'[x_1 := \nu]_{\mathtt{w}}, X)$ by the induction hypothesis, and rule ($\overset{\alpha}{\equiv}$**.6**)

$= \mathbf{Trap}(\lambda x_1.e_1', X)$ by definition of $\mathbf{Trap}$

$= \mathbf{Trap}(e_1, X)$ by assumption

□

Proposition ($\mathbf{Trap}(.,.)$.1): If $\nu \notin X$, then

(a) $\mathbf{Trap}(e, X)[\nu := y]_{\mathtt{w}} \overset{\alpha}{\equiv} \mathbf{Trap}(e[\nu := y]_{\mathtt{w}}, X)$

(b) $\mathbf{Trap}(\lambda\nu.e, X) \overset{\alpha}{\equiv} \lambda\nu.\mathbf{Trap}(e, X)$

Proof of Proposition ($\mathbf{Trap}(.,.)$.1): First observe that (**b**) follows directly from (**a**). Since

$\mathbf{Trap}(\lambda\nu.e, X) = \lambda\nu'.\mathbf{Trap}(e[\nu := \nu']_{\mathtt{w}}, X)$ by assumption

$\overset{\alpha}{\equiv} \lambda\nu'.\mathbf{Trap}(e, X)[\nu := \nu']_{\mathtt{w}}$ by (**a**) and rule ($\overset{\alpha}{\equiv}$**.6**)

$\overset{\alpha}{\equiv} \lambda\nu.\mathbf{Trap}(e, X)$ by proposition (.[.]$_{\mathtt{w}}$**.1**)

The proof of (**a**) is a simple induction on $\rho(e)$.

Base Case: The case when $e \in \mathbb{A}$ is trivial, in the other $e \in \mathbb{X}$ and consequently

$\mathbf{Trap}(e, X)[\nu := y]_{\mathtt{w}} = e[\nu := y]_{\mathtt{w}}$

$\mathbf{Trap}(e[\nu := y]_{\mathtt{w}}, X) = e[\nu := y]_{\mathtt{w}}$ since $e[\nu := y]_{\mathtt{w}} \in \mathbb{X}$

Induction Step Assume the statement holds for expressions of lesser rank. We consider the three cases, depending on the construction of e.

(1. $e = \mathtt{app}(e_0, e_1)$**)**

$$\begin{aligned}
&\mathbf{Trap}(e, X)[\nu := y]_{\mathtt{w}} = \mathbf{Trap}(\mathtt{app}(e_0, e_1), X)[\nu := y]_{\mathtt{w}} && \text{by assumption}\\
&\quad = \mathtt{app}(\mathbf{Trap}(e_0, X), \mathbf{Trap}(e_1, X))[\nu := y]_{\mathtt{w}} && \text{by definition of } \mathbf{Trap}(.,.)\\
&\quad = \mathtt{app}(\mathbf{Trap}(e_0, X)[\nu := y]_{\mathtt{w}}, \mathbf{Trap}(e_1, X)[\nu := y]_{\mathtt{w}}) && \text{by definition of } .[.]_{\mathtt{w}}\\
&\quad \overset{\alpha}{\equiv} \mathtt{app}(\mathbf{Trap}(e_0[\nu := y]_{\mathtt{w}}, X), \mathbf{Trap}(e_1[\nu := y]_{\mathtt{w}}, X)) && \text{by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.2})\\
&\quad = \mathbf{Trap}(\mathtt{app}(e_0[\nu := y]_{\mathtt{w}}, e_1[\nu := y]_{\mathtt{w}}), X) && \text{by definition of } \mathbf{Trap}(.,.)\\
&\quad = \mathbf{Trap}(\mathtt{app}(e_0, e_1)[\nu := y]_{\mathtt{w}}, X) && \text{by definition of } .[.]_{\mathtt{w}}\\
&\quad = \mathbf{Trap}(e[\nu := y]_{\mathtt{w}}, X) && \text{by assumption}
\end{aligned}$$

(2. $e = \bullet^{\sigma}$**)** Now

$$\mathbf{Trap}(e, X)[\nu := y]_{\mathtt{w}} = \mathbf{Trap}(\bullet^{\sigma}, X)[\nu := y]_{\mathtt{w}} = \bullet^{\mathbf{Trap}(\sigma, X)[\nu := y]_{\mathtt{w}}}$$

$$\mathbf{Trap}(e[\nu := y]_{\mathtt{w}}, X) = \mathbf{Trap}(\bullet^{\sigma}[\nu := y]_{\mathtt{w}}, X) = \mathbf{Trap}(\bullet^{\sigma[\nu := y]_{\mathtt{w}}}, X) = \bullet^{\mathbf{Trap}(\sigma[\nu := y]_{\mathtt{w}}, X)}$$

Thus the problem reduces to showing that for every $x \in \mathrm{Dom}(\sigma)$

$$\mathbf{Trap}(\sigma, X)[\nu := y]_{\mathtt{w}}(x) \overset{\alpha}{\equiv} \mathbf{Trap}(\sigma[\nu := y]_{\mathtt{w}}, X)(x)$$

and this is a simple application of the induction hypothesis.

(3. $e = \lambda x.e_0$**)**

$$\begin{aligned}
&\mathbf{Trap}(e, X)[\nu := y]_{\mathtt{w}} = \mathbf{Trap}(\lambda x.e_0, X)[\nu := y]_{\mathtt{w}} && \text{by assumption}\\
&\quad = \lambda\nu'.\mathbf{Trap}(e_0[x := \nu']_{\mathtt{w}}, X)[\nu := y]_{\mathtt{w}} && \text{by the definition of } \mathbf{Trap}(.,.) \text{ for } \nu' \text{ fresh}\\
&\quad = \lambda\nu''.(\mathbf{Trap}(e_0[x := \nu']_{\mathtt{w}}, X)[\nu'' := \nu']_{\mathtt{w}}[\nu := y]_{\mathtt{w}}) && \text{by definition of } .[.]_{\mathtt{w}} \text{ for } \nu'' \text{ fresh}\\
&\quad \overset{\alpha}{\equiv} \lambda\nu''.(\mathbf{Trap}(e_0[x := \nu']_{\mathtt{w}}, X)[\nu := y]_{\mathtt{w}}[\nu'' := \nu']_{\mathtt{w}}) && \text{by proposition } (.[.]_{\mathtt{w}}\textbf{.1}) \text{ and rule } (\overset{\alpha}{\equiv}\textbf{.6})\\
&\quad \overset{\alpha}{\equiv} \lambda\nu''.(\mathbf{Trap}(e_0[x := \nu']_{\mathtt{w}}[\nu := y]_{\mathtt{w}}, X)[\nu'' := \nu']_{\mathtt{w}})\\
&\qquad \text{by the induction hypothesis, proposition } (\overset{\alpha}{\equiv}\textbf{.1}) \text{ and rule } (\overset{\alpha}{\equiv}\textbf{.6})\\
&\quad \overset{\alpha}{\equiv} \lambda\nu''.\mathbf{Trap}(e_0[x := \nu']_{\mathtt{w}}[\nu := y]_{\mathtt{w}}[\nu'' := \nu']_{\mathtt{w}}, X) && \text{again by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.6})\\
&\quad = \mathbf{Trap}(\lambda\nu'.e_0[x := \nu']_{\mathtt{w}}[\nu := y]_{\mathtt{w}}, X) && \text{by the definition of } \mathbf{Trap}(.,.)\\
&\quad = \mathbf{Trap}((\lambda x.e_0)[\nu := y]_{\mathtt{w}}, X) && \text{by definition of } .[.]_{\mathtt{w}}\\
&\quad = \mathbf{Trap}(e[\nu := y]_{\mathtt{w}}, X)
\end{aligned}$$

□

Proposition ($\overset{\alpha}{\equiv}$**.3):**

$$(e_0 \overset{\alpha}{\equiv} e_1 \ \wedge\ e_0' \overset{\alpha}{\equiv} e_1') \ \Rightarrow\ e_0[\![e_0']\!] \overset{\alpha}{\equiv} e_1[\![e_1']\!]$$

Proof of Proposition ($\overset{\alpha}{\equiv}$**.3):** As usual the proof is a simple induction on $\rho(e_0) = \rho(e_1)$.

Base Case: In this case $e_i \in \mathbb{A} \cup \mathbb{X}$ and by hypothesis $e_0 = e_1$ consequently by computation:

$$e_0[\![e_0']\!] = e_0 = e_1 = e_1[\![e_1']\!]$$

Induction Step Assume the statement holds for expressions of lesser rank. We consider the three cases, depending on the construction of e_i.

(1. $e_i = \mathtt{app}(e_i', e_i'')$**)** By computation:

$$\begin{aligned}
e_0[\![e_0']\!] &= \mathtt{app}(e_0', e_0'')[\![e_0']\!] && \text{by assumption}\\
&= \mathtt{app}(e_0'[\![e_0']\!], e_0''[\![e_0']\!]) && \text{by definition of } .[\![.]\!]\\
&\overset{\alpha}{\equiv} \mathtt{app}(e_1'[\![e_1']\!], e_1''[\![e_1']\!]) && \text{by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.2})\\
&= \mathtt{app}(e_1', e_1'')[\![e_1']\!] && \text{by definition of } .[\![.]\!]\\
&= e_1[\![e_1']\!] && \text{by assumption}
\end{aligned}$$

(2. $e_i = \bullet^{\sigma_i}$**)** The assumption that $e_0 \overset{\alpha}{\equiv} e_1$ implies that $\mathrm{Dom}(\sigma_0) = \mathrm{Dom}(\sigma_1)$. Furthermore by the induction hypothesis and rules ($\overset{\alpha}{\equiv}$**.3**) & ($\overset{\alpha}{\equiv}$**.4**) we have that

$$(\forall x \in \mathrm{Dom}(\sigma_i))(\sigma_0(x)[\![e_0']\!] \overset{\alpha}{\equiv} \sigma_1(x)[\![e_1']\!])$$

Thus

$$\sigma_0[\![e_0']\!] \overset{\alpha}{\equiv} \sigma_1[\![e_1']\!]$$

Now by computation:

$$\begin{aligned}
e_0[\![e_0']\!] &= \bullet^{\sigma_0}[\![e_0']\!] && \text{by assumption}\\
&= \mathbf{Trap}(e_0', \mathrm{Dom}(\sigma_0))[\sigma_0[\![e_0']\!]]_{\mathtt{w}} && \text{by definition of } .[\![.]\!]\\
&= \mathbf{Trap}(e_0', \mathrm{Dom}(\sigma_1))[\sigma_0[\![e_0']\!]]_{\mathtt{w}} && \text{since } \mathrm{Dom}(\sigma_0) = \mathrm{Dom}(\sigma_1)\\
&\overset{\alpha}{\equiv} \mathbf{Trap}(e_1', \mathrm{Dom}(\sigma_1))[\sigma_0[\![e_0']\!]]_{\mathtt{w}} && \text{by proposition } (\overset{\alpha}{\equiv}\textbf{.2}) \text{ and the initial hypothesis}\\
&\overset{\alpha}{\equiv} \mathbf{Trap}(e_1', \mathrm{Dom}(\sigma_1))[\sigma_1[\![e_1']\!]]_{\mathtt{w}} && \text{by proposition } (\overset{\alpha}{\equiv}\textbf{.1}) \text{ and the initial observations}\\
&= \bullet^{\sigma_1}[\![e_1']\!] && \text{by definition of } .[\![.]\!]\\
&= e_1[\![e_1']\!] && \text{by assumption}
\end{aligned}$$

(3. $e_i = \lambda x_i.e_i''$**)** By computation:

$$\begin{aligned}
e_0[\![e_0']\!] &= (\lambda x_0.e_0'')[\![e_0']\!] && \text{by assumption}\\
&= \lambda\nu.(e_0''[x_0 := \nu]_{\mathtt{w}}[\![e_0']\!]) && \text{by definition of } .[\![.]\!] \text{ for } \nu \text{ fresh}\\
&\overset{\alpha}{\equiv} \lambda\nu.(e_1''[x_1 := \nu]_{\mathtt{w}}[\![e_1']\!] && \text{by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.6})\\
&= (\lambda x_1.e_1'')[\![e_1']\!] && \text{by definition of } .[\![.]\!]\\
&= e_1[\![e_1']\!] && \text{by assumption}
\end{aligned}$$

□

Lemma (.⟦.⟧.1):

(a) $\nu \notin \mathrm{FV}(e_0) \;\Rightarrow\; e[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket \overset{\alpha}{\equiv} e\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}}$

(b) $(\lambda\nu.e)\llbracket e_0 \rrbracket \overset{\alpha}{\equiv} \lambda\nu.(e\llbracket e_0 \rrbracket)$ if $\nu \notin \mathrm{FV}(e_0)$

Proof of Lemma (.⟦.⟧.1): First observe that (**b**) follows directly from (**a**). Since

$$\begin{aligned}(\lambda\nu.e)\llbracket e_0 \rrbracket &= \lambda\nu'.(e[\nu := \nu']_{\mathtt{w}}\llbracket e_0 \rrbracket) && \text{by definition of } .\llbracket . \rrbracket \\ &\overset{\alpha}{\equiv} \lambda\nu'.(e\llbracket e_0 \rrbracket[\nu := \nu']_{\mathtt{w}}) && \text{by (a) and rule } (\overset{\alpha}{\equiv}\textbf{.6}) \\ &\overset{\alpha}{\equiv} \lambda\nu.(e\llbracket e_0 \rrbracket) && \text{by proposition } (.[.]_{\mathtt{w}}\textbf{.1})\end{aligned}$$

Base Case: The atomic case is immediate, in the other case $e \in \mathbb{X}$ so consequently

$$e[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket = e[\nu := y]_{\mathtt{w}} = e\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}} \qquad \text{since } e[\nu := y]_{\mathtt{w}} \in \mathbb{X}$$

Induction Step Assume the statement holds for expressions of lesser rank. We consider the three cases, depending on the construction of e.

(1. $e = \mathtt{app}(e_0, e_1)$)

$$\begin{aligned}e[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket &= \mathtt{app}(e_0, e_1)[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket && \text{by assumption} \\ &= \mathtt{app}(e_0[\nu := y]_{\mathtt{w}}, e_1[\nu := y]_{\mathtt{w}})\llbracket e_0 \rrbracket && \text{by definition of } .[.]_{\mathtt{w}} \\ &= \mathtt{app}(e_0[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket, e_1[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket) && \text{by definition of } .\llbracket . \rrbracket \\ &= \mathtt{app}(e_0\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}}, e_1\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}}) && \text{by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.2}) \\ &= \mathtt{app}(e_0\llbracket e_0 \rrbracket, e_1\llbracket e_0 \rrbracket)[\nu := y]_{\mathtt{w}} && \text{by definition of } .[.]_{\mathtt{w}} \\ &= \mathtt{app}(e_0, e_1)\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}} && \text{by definition of } .\llbracket . \rrbracket \\ &= e\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}} && \text{by assumption}\end{aligned}$$

(2. $e = \bullet^{\sigma}$)

$$\begin{aligned}e[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket &= \bullet^{\sigma}[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket && \text{by assumption} \\ &= \bullet^{\sigma[\nu := y]_{\mathtt{w}}}\llbracket e_0 \rrbracket && \text{by definition of } .[.]_{\mathtt{w}} \\ &= \mathbf{Trap}(e_0, \mathrm{Dom}(\sigma[\nu := y]_{\mathtt{w}}))[\sigma[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket]_{\mathtt{w}} && \text{by definition of } .\llbracket . \rrbracket \\ &= \mathbf{Trap}(e_0, \mathrm{Dom}(\sigma))[\sigma[\nu := y]_{\mathtt{w}}\llbracket e_0 \rrbracket]_{\mathtt{w}} && \text{since } \mathrm{Dom}(\sigma[\nu := y]_{\mathtt{w}}) = \mathrm{Dom}(\sigma)\end{aligned}$$

On the other hand

$$\begin{aligned}e\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}} &= \bullet^{\sigma}\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}} && \text{by assumption} \\ &= \mathbf{Trap}(e_0, \mathrm{Dom}(\sigma))[\sigma\llbracket e_0 \rrbracket]_{\mathtt{w}}[\nu := y]_{\mathtt{w}} && \text{by definition of } .\llbracket . \rrbracket\end{aligned}$$

Observe that the conditions of proposition (.[.]$_{\mathtt{w}}$**.2**) are met since $\nu \in \mathrm{FV}(\mathbf{Trap}(e_0, \mathrm{Dom}(\sigma)))$ iff $\nu \in \mathrm{Dom}(\sigma)$ because by assumption $\nu \notin \mathrm{FV}(e_0)$. Thus

$$\mathbf{Trap}(e_0, \mathrm{Dom}(\sigma))[\sigma\llbracket e_0 \rrbracket]_{\mathtt{w}}[\nu := y]_{\mathtt{w}} = \mathbf{Trap}(e_0, \mathrm{Dom}(\sigma))[\sigma\llbracket e_0 \rrbracket[\nu := y]_{\mathtt{w}}]_{\mathtt{w}} \qquad \text{by proposition } (.[.]_{\mathtt{w}}\textbf{.2})$$

Thus we are reduced to showing that

$$\sigma[\nu := y]_{\mathtt{w}}[\![e_0]\!] \overset{\alpha}{\equiv} \sigma[\![e_0]\!][\nu := y]_{\mathtt{w}}$$

and this again is a simple pointwise application of the induction hypothesis.

(3. $e = \lambda x.e'$**)**

$$\begin{aligned}
e[\nu := y]_{\mathtt{w}}[\![e_0]\!] &= (\lambda x.e')[\nu := y]_{\mathtt{w}}[\![e_0]\!] && \text{by assumption}\\
&= (\lambda \nu'.e'[x := \nu']_{\mathtt{w}}[\nu := y]_{\mathtt{w}})[\![e_0]\!] && \text{by definition of } .[.]_{\mathtt{w}} \text{ for } \nu' \text{ fresh}\\
&= \lambda \nu''.(e'[x := \nu']_{\mathtt{w}}[\nu := y]_{\mathtt{w}}[\nu' := \nu'']_{\mathtt{w}}[\![e_0]\!]) && \text{by definition of } .[\![.]\!] \text{ for } \nu'' \text{ fresh}\\
&= \lambda \nu''.(e'[x := \nu']_{\mathtt{w}}[\nu := y]_{\mathtt{w}}[\![e_0]\!][\nu' := \nu'']_{\mathtt{w}}) && \text{by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.6})\\
&= \lambda \nu''.(e'[x := \nu']_{\mathtt{w}}[\![e_0]\!][\nu := y]_{\mathtt{w}}[\nu' := \nu'']_{\mathtt{w}}) && \text{by the induction hypothesis, rule } (\overset{\alpha}{\equiv}\textbf{.6}) \text{ and proposition } (\overset{\alpha}{\equiv}\textbf{.1})\\
&= \lambda \nu''.(e'[x := \nu']_{\mathtt{w}}[\![e_0]\!][\nu' := \nu'']_{\mathtt{w}}[\nu := y]_{\mathtt{w}}) && \text{by the induction hypothesis, rule } (\overset{\alpha}{\equiv}\textbf{.6}) \text{ and proposition } (.[.]_{\mathtt{w}}\textbf{.1})\\
&= (\lambda \nu'.e'[x := \nu']_{\mathtt{w}}[\![e_0]\!])[\nu := y]_{\mathtt{w}} && \text{by definition of } .[.]_{\mathtt{w}}\\
&= (\lambda x.e')[\![e_0]\!][\nu := y]_{\mathtt{w}} && \text{by definition of } .[\![.]\!]\\
&= e[\![e_0]\!][\nu := y]_{\mathtt{w}} && \text{by assumption}
\end{aligned}$$

□

5.3 The Proofs of Claims in §2.4

Lemma ($.[.]_{\mathtt{s}}$**.1):**

$$e[\sigma]_{\mathtt{s}} \overset{\alpha}{\equiv} \mathbf{Trap}(e, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}$$

Proof of Lemma ($.[.]_{\mathtt{s}}$**.1):** The proof is a simple induction on $\rho(e)$.

Base Case: In this case $e \in \mathbb{A} \cup \mathbb{X}$. The atomic case is trivial, consequently we do the case when $e \in \mathbb{X}$:

$$e[\sigma]_{\mathtt{s}} = \begin{cases} \sigma(e) & \text{if } e \in \mathrm{Dom}(\sigma),\\ e & \text{otherwise} \end{cases}$$

$$\mathbf{Trap}(e, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} = e[\sigma]_{\mathtt{w}} = \begin{cases} \sigma(e) & \text{if } e \in \mathrm{Dom}(\sigma),\\ e & \text{otherwise} \end{cases}$$

Induction Step Assume the statement holds for expressions of lesser rank. We consider the three cases, depending on the construction of e.

(1. $e = \mathtt{app}(e_0, e_1)$**)** By computation:

$$\begin{aligned}
e[\sigma]_{\mathtt{s}} &= \mathtt{app}(e_0, e_1)[\sigma]_{\mathtt{s}} && \text{by assumption}\\
&= \mathtt{app}(e_0[\sigma]_{\mathtt{s}}, e_1[\sigma]_{\mathtt{s}}) && \text{by definition of } .[.]_{\mathtt{s}}\\
&\overset{\alpha}{\equiv} \mathtt{app}(\mathbf{Trap}(e_0, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}, \mathbf{Trap}(e_1, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}) && \text{by the induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.2})\\
&= \mathbf{Trap}(\mathtt{app}(e_0, e_1), \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} && \text{by definition of } \mathbf{Trap}\\
&= \mathbf{Trap}(e, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} && \text{by assumption}
\end{aligned}$$

(2. $e = \bullet^{\sigma_0}$) First observe that

$$e[\sigma]_{\mathtt{s}} = \bullet^{\sigma_0}[\sigma]_{\mathtt{s}} = \bullet^{\sigma_0[\sigma]_{\mathtt{s}}}$$

$$\mathbf{Trap}(e, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} = \mathbf{Trap}(\bullet^{\sigma_0}, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} = \bullet^{\mathbf{Trap}(\sigma_0, \mathrm{Dom}(\sigma))}[\sigma]_{\mathtt{w}} = \bullet^{\mathbf{Trap}(\sigma_0, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}}$$

So it suffices to show that

$$\sigma_0[\sigma]_{\mathtt{s}} \stackrel{\alpha}{\equiv} \mathbf{Trap}(\sigma_0, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}$$

Now also observe that

$$\mathbf{Trap}(\sigma_0, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}$$
$$= \underline{\lambda} y \in \mathrm{Dom}(\sigma_0) \cup \mathrm{Dom}(\sigma). \begin{cases} \mathbf{Trap}(\sigma_0(y), \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} & \text{if } y \in \mathrm{Dom}(\sigma_0) \\ y[\sigma]_{\mathtt{w}} & \text{if } y \in \mathrm{Dom}(\sigma) - \mathrm{Dom}(\sigma_0) \end{cases}$$

Thus

$$\mathrm{Dom}(\sigma_0[\sigma]_{\mathtt{s}}) = \mathrm{Dom}(\sigma_0) \cup \mathrm{Dom}(\sigma) = \mathrm{Dom}(\mathbf{Trap}(\sigma_0, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}})$$

So it suffices to show that for any $x \in \mathrm{Dom}(\sigma_0) \cup \mathrm{Dom}(\sigma)$ that

$$\sigma_0[\sigma]_{\mathtt{s}}(x) \stackrel{\alpha}{\equiv} \mathbf{Trap}(\sigma_0, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}(x)$$

The case when $x \in \mathrm{Dom}(\sigma) - \mathrm{Dom}(\sigma_0)$ is immediate. If $x \in \mathrm{Dom}(\sigma_0)$, then by computation

$$\sigma_0[\sigma]_{\mathtt{s}}(x) = \sigma_0(x)[\sigma]_{\mathtt{s}}$$
$$\stackrel{\alpha}{\equiv} \mathbf{Trap}(\sigma_0(x), \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} \qquad \text{by the induction hypothesis}$$
$$= \mathbf{Trap}(\sigma_0, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}(x)$$

(3. $e = \lambda x.e_0$) By computation:

$$e[\sigma]_{\mathtt{s}} = (\lambda x.e_0)[\sigma]_{\mathtt{s}}$$
$$= \lambda \nu.(e_0[x := \nu]_{\mathtt{w}}[\sigma]_{\mathtt{s}}) \qquad \text{by definition of } .[.]_{\mathtt{s}}$$
$$\stackrel{\alpha}{\equiv} \lambda \nu.(\mathbf{Trap}(e_0[x := \nu]_{\mathtt{w}}, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}) \qquad \text{by the induction hypothesis}$$
$$\stackrel{\alpha}{\equiv} (\lambda \nu.\mathbf{Trap}(e_0[x := \nu]_{\mathtt{w}}, \mathrm{Dom}(\sigma)))[\sigma]_{\mathtt{w}} \qquad \text{by proposition } (.[.]_{\mathtt{w}}\textbf{.1}) \text{ since } \nu \text{ is fresh}$$
$$= \mathbf{Trap}(e, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} \qquad \text{by definition of } \mathbf{Trap}.$$

□

Lemma ($.[.]_{\mathtt{s}}$.2):

(a) $(\nu_0 \notin (\mathrm{FV}(\sigma) \cup \mathrm{Dom}(\sigma)) \wedge \nu_1 \notin \mathrm{Dom}(\sigma)) \Rightarrow e[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{s}} \stackrel{\alpha}{\equiv} e[\sigma]_{\mathtt{s}}[\nu_0 := \nu_1]_{\mathtt{w}}$

(b) $(\lambda \nu.e)[\sigma]_{\mathtt{s}} \stackrel{\alpha}{\equiv} \lambda \nu.(e[\sigma]_{\mathtt{s}}) \qquad$ if $\nu \notin (\mathrm{FV}(\sigma) \cup \mathrm{Dom}(\sigma))$

Proof of Lemma ($.[.]_{\mathtt{s}}$.2): First observe that (**b**) follows directly from (**a**). Since

$$(\lambda \nu.e)[\sigma]_{\mathtt{s}} = \lambda \nu'.(e[\nu := \nu']_{\mathtt{w}}[\sigma]_{\mathtt{s}}) \qquad \text{by definition of } .[.]_{\mathtt{s}} \text{ for } \nu' \text{ fresh}$$
$$\stackrel{\alpha}{\equiv} \lambda \nu'.(e[\sigma]_{\mathtt{s}}[\nu := \nu']_{\mathtt{w}}) \qquad \text{by part (a) and rule } (\stackrel{\alpha}{\equiv}\textbf{.6})$$
$$\stackrel{\alpha}{\equiv} \lambda \nu.(e[\sigma]_{\mathtt{s}}) \qquad \text{by proposition } (.[.]_{\mathtt{w}}\textbf{.1})$$

To prove (**a**) observe that

$$
\begin{aligned}
e[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{s}} &= \mathbf{Trap}(e[\nu_0 := \nu_1]_{\mathtt{w}}, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}} && \text{by lemma } (.[.]_{\mathtt{s}}\textbf{.1}) \\
&\overset{\alpha}{\equiv} \mathbf{Trap}(e, \mathrm{Dom}(\sigma))[\nu_0 := \nu_1]_{\mathtt{w}}[\sigma]_{\mathtt{w}} && \text{by proposition } (\mathbf{Trap}(.,.)\textbf{.1}), \text{ since } \nu_0 \notin \mathrm{Dom}(\sigma) \\
&\overset{\alpha}{\equiv} \mathbf{Trap}(e, \mathrm{Dom}(\sigma))[\sigma]_{\mathtt{w}}[\nu_0 := \nu_1]_{\mathtt{w}} && \text{by proposition } (.[.]_{\mathtt{w}}\textbf{.1}) = e[\sigma]_{\mathtt{s}}[\nu_0 := \nu_1]_{\mathtt{w}} \quad \text{by lemma } (.[.]_{\mathtt{s}}\textbf{.1})
\end{aligned}
$$

□

Proposition $(.[.]_{\mathtt{s}}.3)$**:**

$$e[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} \overset{\alpha}{\equiv} e[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}$$

Proof of Proposition $(.[.]_{\mathtt{s}}.3)$**:** The proof is a simple induction on $\rho(e)$.

Base Case: We omit the simple atomic case. Assume that $e \in \mathbb{X}$ consequently

$$
\begin{aligned}
e[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} &= \begin{cases} \sigma_0(e)[\sigma_1]_{\mathtt{s}} & \text{if } e \in \mathrm{Dom}(\sigma_0) \\ e[\sigma_1]_{\mathtt{s}} & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma_0(e)[\sigma_1]_{\mathtt{s}} & \text{if } e \in \mathrm{Dom}(\sigma_0) \\ \sigma_1(e) & \text{if } e \in \mathrm{Dom}(\sigma_1) - \mathrm{Dom}(\sigma_0) \\ e & \text{otherwise} \end{cases} \\
&= e[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}
\end{aligned}
$$

Induction Step Assume the statement holds for expressions of lesser rank. We consider the three cases, depending on the construction of e.

(1. $e = \mathtt{app}(e_0, e_1)$**)** In this case:

$$
\begin{aligned}
e[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} &= \mathtt{app}(e_0, e_1)[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} && \text{by assumption} \\
&= \mathtt{app}(e_0[\sigma_0]_{\mathtt{s}}, e_1[\sigma_0]_{\mathtt{s}})[\sigma_1]_{\mathtt{s}} && \text{by definition of } .[.]_{\mathtt{s}} \\
&= \mathtt{app}(e_0[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}}, e_1[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}}) && \text{again by definition of } .[.]_{\mathtt{s}} \\
&\overset{\alpha}{\equiv} \mathtt{app}(e_0[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}, e_1[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}) && \text{by the induction hypothesis} \\
&= \mathtt{app}(e_0, e_1)[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} && \text{again by definition of } .[.]_{\mathtt{s}} \\
&= e[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} && \text{by assumption}
\end{aligned}
$$

(2. $e = \bullet^{\sigma}$**)** In this situation note that:

$$
\begin{aligned}
e[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} &= \bullet^{\sigma}[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} = \bullet^{\sigma[\sigma_0]_{\mathtt{s}}}[\sigma_1]_{\mathtt{s}} = \bullet^{\sigma[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}}} \\
e[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} &= \bullet^{\sigma}[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} = \bullet^{\sigma[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}}
\end{aligned}
$$

So it suffice to show that

$$\sigma[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} \overset{\alpha}{\equiv} \sigma[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}$$

To this end note that

$$\mathrm{Dom}(\sigma[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}}) = \mathrm{Dom}(\sigma) \cup \mathrm{Dom}(\sigma_0) \cup \mathrm{Dom}(\sigma_1) = \mathrm{Dom}(\sigma[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}})$$

Thus we need only demonstrate that for every $x \in \mathrm{Dom}(\sigma) \cup \mathrm{Dom}(\sigma_0) \cup \mathrm{Dom}(\sigma_1)$

$$\sigma[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}}(x) \stackrel{\alpha}{\equiv} \sigma[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}(x)$$

Now

$$\begin{aligned}
&\sigma[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}}(x) \\
&\quad = \begin{cases} \sigma(x)[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} & \text{if } x \in \mathrm{Dom}(\sigma) \\ \sigma_0(x)[\sigma_1]_{\mathtt{s}} & \text{if } x \in \mathrm{Dom}(\sigma_0) - \mathrm{Dom}(\sigma) \\ \sigma_1(x) & \text{if } x \in \mathrm{Dom}(\sigma_1) - (\mathrm{Dom}(\sigma) \cup \mathrm{Dom}(\sigma_0)) \end{cases} \\
&\quad = \begin{cases} \sigma(x)[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} & \text{if } x \in \mathrm{Dom}(\sigma) \\ \sigma_0(x)[\sigma_1]_{\mathtt{s}} & \text{if } x \in \mathrm{Dom}(\sigma_0) - \mathrm{Dom}(\sigma) \\ \sigma_1(x) & \text{if } x \in \mathrm{Dom}(\sigma_1) - (\mathrm{Dom}(\sigma) \cup \mathrm{Dom}(\sigma_0)) \end{cases} \\
&\qquad \text{by the induction hypothesis} \\
&\quad = \begin{cases} \sigma(x)[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} & \text{if } x \in \mathrm{Dom}(\sigma) \\ \sigma[\sigma_1]_{\mathtt{s}}(x) & \text{if } x \in \mathrm{Dom}(\sigma_0[\sigma_1]_{\mathtt{s}}) - \mathrm{Dom}(\sigma) \end{cases} \\
&\qquad \text{by definition of } \sigma[\sigma_1]_{\mathtt{s}} \\
&\quad = \sigma[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}(x)
\end{aligned}$$

(3. $e = \lambda x.e_0$**)**

$$\begin{aligned}
e[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} &= (\lambda x.e_0)[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}} && \text{by assumption} \\
&= (\lambda \nu.e_0[x := \nu]_{\mathtt{w}}[\sigma_0]_{\mathtt{s}})[\sigma_1]_{\mathtt{s}} && \text{by definition of } .[.]_{\mathtt{s}} \text{ for } \nu \text{ fresh} \\
&\stackrel{\alpha}{\equiv} \lambda \nu.(e_0[x := \nu]_{\mathtt{w}}[\sigma_0]_{\mathtt{s}}[\sigma_1]_{\mathtt{s}}) && \text{by definition of } .[.]_{\mathtt{s}} \text{ since } \nu \text{ is fresh and lemma } (.[.]_{\mathtt{s}}\textbf{.2}) \\
&\stackrel{\alpha}{\equiv} \lambda \nu.(e_0[x := \nu]_{\mathtt{w}}[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}}) && \text{by the induction hypothesis and rule } (\stackrel{\alpha}{\equiv}\textbf{.6}) \\
&\stackrel{\alpha}{\equiv} (\lambda \nu.e_0[x := \nu]_{\mathtt{w}})[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} && \text{by definition of } .[.]_{\mathtt{s}} \text{ since } \nu \text{ is fresh and lemma } (.[.]_{\mathtt{s}}\textbf{.2}) \\
&\stackrel{\alpha}{\equiv} (\lambda x.e_0)[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} && \text{by proposition } (.[.]_{\mathtt{w}}\textbf{.1}) \\
&= e[\sigma_0[\sigma_1]_{\mathtt{s}}]_{\mathtt{s}} && \text{by assumption}
\end{aligned}$$

□

Proof of Theorem (2): The proof is by induction on $\rho(e)$.

Base Case: As usual we omit the atomic case. In the other case $e \in \mathbb{X}$ consequently

$$\begin{aligned}
&e[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}} = e[\sigma[\![e']\!]]_{\mathtt{s}} \\
&\quad = \begin{cases} \sigma[\![e']\!](e) & \text{if } e \in \mathrm{Dom}(\sigma[\![e']\!]) = \mathrm{Dom}(\sigma) \\ e & \text{otherwise} \end{cases} \\
&\quad = \begin{cases} \sigma[\![e']\!](e) & \text{if } e \in \mathrm{Dom}(\sigma) \\ e[\![e']\!] & \text{otherwise} \end{cases} \\
&\quad = \begin{cases} \sigma(e)[\![e']\!] & \text{if } e \in \mathrm{Dom}(\sigma) \\ e[\![e']\!] & \text{otherwise} \end{cases} \\
&\quad = e[\sigma]_{\mathtt{s}}[\![e']\!]
\end{aligned}$$

Induction Step Assume the statement holds for expressions of lesser rank. We consider the three cases, depending on the construction of e.

(1. $e = \mathtt{app}(e_0, e_1)$**)** In this case

$$
\begin{aligned}
e[\sigma]_{\mathtt{s}}[\![e']\!] &= \mathtt{app}(e_0, e_1)[\sigma]_{\mathtt{s}}[\![e']\!] && \text{by assumption}\\
&= \mathtt{app}(e_0[\sigma]_{\mathtt{s}}, e_1[\sigma]_{\mathtt{s}})[\![e']\!] && \text{by definition of } .[.]_{\mathtt{s}}\\
&= \mathtt{app}(e_0[\sigma]_{\mathtt{s}}[\![e']\!], e_1[\sigma]_{\mathtt{s}}[\![e']\!]) && \text{by definition of } .[\![.]\!]\\
&\overset{\alpha}{\equiv} \mathtt{app}(e_0[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}, e_1[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}) && \text{by induction hypothesis and rule } (\overset{\alpha}{\equiv}\textbf{.2})\\
&= \mathtt{app}(e_0[\![e']\!], e_1[\![e']\!])[\sigma[\![e']\!]]_{\mathtt{s}} && \text{by definition of } .[.]_{\mathtt{s}}\\
&= \mathtt{app}(e_0, e_1)[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}} && \text{by definition of } .[\![.]\!]\\
&= e[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}} && \text{by assumption}
\end{aligned}
$$

(2. $e = \bullet^{\sigma_0}$**)** First note that

$$
\begin{aligned}
e[\sigma]_{\mathtt{s}}[\![e']\!] &= \bullet^{\sigma_0}[\sigma]_{\mathtt{s}}[\![e']\!] && \text{by assumption}\\
&= \bullet^{\sigma_0[\sigma]_{\mathtt{s}}}[\![e']\!] && \text{by definition of } .[.]_{\mathtt{s}}\\
&= \mathbf{Trap}(e', \mathrm{Dom}(\sigma_0[\sigma]_{\mathtt{s}}))[\sigma_0[\sigma]_{\mathtt{s}}[\![e']\!]]_{\mathtt{w}} && \text{by definition of } .[\![.]\!]\\
&\overset{\alpha}{\equiv} e'[\sigma_0[\sigma]_{\mathtt{s}}[\![e']\!]]_{\mathtt{s}} && \text{by lemma } (.[.]_{\mathtt{s}}\textbf{.1})
\end{aligned}
$$

On the other hand

$$e[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}} = \bullet^{\sigma_0}[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}} = e'[\sigma_0[\![e']\!]_{\mathtt{s}}[\sigma[\![e']\!]]_{\mathtt{s}}$$

Thus by composition of strong substitution (proposition $(.[.]_{\mathtt{s}}\textbf{.3})$) it suffices to prove that

$$\sigma_0[\sigma]_{\mathtt{s}}[\![e']\!] = \sigma_0[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}$$

Clearly

$$\mathrm{Dom}(\sigma_0[\sigma]_{\mathtt{s}}[\![e']\!]) = \mathrm{Dom}(\sigma_0) \cup \mathrm{Dom}(\sigma) = \mathrm{Dom}(\sigma_0[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}})$$

Thus it suffices to show that for every $x \in \mathrm{Dom}(\sigma_0) \cup \mathrm{Dom}(\sigma)$

$$\sigma_0[\sigma]_{\mathtt{s}}[\![e']\!](x) = \sigma_0[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}(x)$$

We consider the two cases depending on whether $x \in \mathrm{Dom}(\sigma_0)$ or $x \in \mathrm{Dom}(\sigma) - \mathrm{Dom}(\sigma_0)$. In the former case we have that

$$
\begin{aligned}
\sigma_0[\sigma]_{\mathtt{s}}[\![e']\!](x) &= \sigma_0(x)[\sigma]_{\mathtt{s}}[\![e']\!]\\
&\overset{\alpha}{\equiv} \sigma_0(x)[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}} && \text{by the induction hypothesis}\\
&= \sigma_0[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}(x)
\end{aligned}
$$

In the latter case

$$\sigma_0[\sigma]_{\mathtt{s}}[\![e']\!](x) = \sigma(x)[\![e']\!] = \sigma_0[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}(x)$$

(3. $e = \lambda x.e_0$**)**

$$
\begin{aligned}
e[\sigma]_{\mathtt{s}}[\![e']\!] &= (\lambda x.e_0)[\sigma]_{\mathtt{s}}[\![e']\!] && \text{by assumption}\\
&= (\lambda \nu.e_0[x := \nu]_{\mathtt{w}}[\sigma]_{\mathtt{s}})[\![e']\!] && \text{by definition of } .[.]_{\mathtt{s}} \text{ for } \nu \text{ fresh}
\end{aligned}
$$

$\stackrel{\alpha}{\equiv} \lambda\nu.(e_0[x := \nu]_{\mathtt{w}}[\sigma]_{\mathtt{s}}[\![e']\!])$ since ν is fresh and lemma (.[.].**1**)

$\stackrel{\alpha}{\equiv} \lambda\nu.(e_0[x := \nu]_{\mathtt{w}}[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}})$ by the induction hypothesis and ($\stackrel{\alpha}{\equiv}$**.6**)

$\stackrel{\alpha}{\equiv} (\lambda\nu.e_0[x := \nu]_{\mathtt{w}}[\![e']\!])[\sigma[\![e']\!]]_{\mathtt{s}}$ since ν is fresh and lemma (.[.]$_{\mathtt{s}}$.**2**)

$\stackrel{\alpha}{\equiv} (\lambda\nu.e_0[x := \nu]_{\mathtt{w}})[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}$ since ν is fresh and lemma (.[.].**1**)

$= (\lambda x.e_0)[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}$

$= e[\![e']\!][\sigma[\![e']\!]]_{\mathtt{s}}$ by assumption

$\square_2$

Proposition (.[.]$_{\mathtt{s}}$.4)**:**

$\mathbf{Trap}(e, X) \stackrel{\alpha}{\equiv} e[\underline{\lambda} x \in X.x]_{\mathtt{s}}$

5.4 The Proofs of Claims in §3.1

Lemma (3): If ν is fresh, then

$e[x := v]_{\mathtt{w}}[\![e_0]\!] \stackrel{\alpha}{\equiv} e[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}$

Proof of Lemma (3): The proof is a simple induction on $\rho(e)$.

Base Case: The atomic case is vacuous. In the other case $e \in \mathbb{X}$ consequently we consider two cases. If $e = x$, then

$e[x := v]_{\mathtt{w}}[\![e_0]\!] = v[\![e_0]\!] = \nu[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}} = e[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}$

On the otherhand if $e \neq x$, then

$e[x := v]_{\mathtt{w}}[\![e_0]\!] = e[\![e_0]\!] = e[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}$

since $\nu \notin \mathrm{FV}(e[x := \nu]_{\mathtt{w}}[\![e_0]\!])$.

Induction Step Assume the statement holds for expressions of less rank. We consider the three cases, depending on the construction of e.

(1. $e = \mathtt{app}(e', e'')$**)**

$e[x := v]_{\mathtt{w}}[\![e_0]\!] = \mathtt{app}(e', e'')[x := v]_{\mathtt{w}}[\![e_0]\!]$

$= \mathtt{app}(e'[x := v]_{\mathtt{w}}, e''[x := v]_{\mathtt{w}})[\![e_0]\!]$

$= \mathtt{app}(e'[x := v]_{\mathtt{w}}[\![e_0]\!], e''[x := v]_{\mathtt{w}}[\![e_0]\!])$

$\stackrel{\alpha}{\equiv} \mathtt{app}(e'[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}, e''[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}})$ by the induction hypothesis

$= \mathtt{app}(e'[x := \nu]_{\mathtt{w}}[\![e_0]\!], e''[x := \nu]_{\mathtt{w}})[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}$

$= \mathtt{app}(e'[x := \nu]_{\mathtt{w}}, e''[x := \nu]_{\mathtt{w}})[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}$

$= \mathtt{app}(e', e'')[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}$

$= e[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}$

(2. $e = \bullet^{\sigma}$**)** In this case we have by computation

$$\begin{aligned}
e[x := v]_{\mathtt{w}}[\![e_0]\!] &= (\bullet^{\sigma})[x := v]_{\mathtt{w}}[\![e_0]\!] \\
&= (\bullet^{\sigma[x:=v]_{\mathtt{w}}})[\![e_0]\!] \\
&= \mathbf{Trap}(e_0, \mathrm{Dom}(\sigma[x := v]_{\mathtt{w}}))[\sigma[x := v]_{\mathtt{w}}[\![e_0]\!]]_{\mathtt{w}} \\
&= \mathbf{Trap}(e_0, \mathrm{Dom}(\sigma))[\sigma[x := v]_{\mathtt{w}}[\![e_0]\!]]_{\mathtt{w}}
\end{aligned}$$

Similarly:

$$\begin{aligned}
e[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}} &= (\bullet^{\sigma})[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}} \\
&= (\bullet^{\sigma[x:=\nu]_{\mathtt{w}}})[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}} \\
&= (\mathbf{Trap}(e_0, \mathrm{Dom}(\sigma[x := \nu]_{\mathtt{w}}))[\sigma[x := \nu]_{\mathtt{w}}[\![e_0]\!]]_{\mathtt{w}})[\nu := v[\![e_0]\!]]_{\mathtt{w}} \\
&= (\mathbf{Trap}(e_0, \mathrm{Dom}(\sigma))[\sigma[x := \nu]_{\mathtt{w}}[\![e_0]\!]]_{\mathtt{w}})[\nu := v[\![e_0]\!]]_{\mathtt{w}}
\end{aligned}$$

So letting $e' = \mathbf{Trap}(e_0, \mathrm{Dom}(\sigma))$ it suffices to show that

$$e'[\sigma[x := v]_{\mathtt{w}}[\![e_0]\!]]_{\mathtt{w}} \stackrel{\alpha}{\equiv} e'[\sigma[x := \nu]_{\mathtt{w}}[\![e_0]\!]]_{\mathtt{w}}[\nu := v[\![e_0]\!]]_{\mathtt{w}}$$

Notice that by definition $\nu \notin \mathrm{FV}(e')$. Thus by lemma (.$[.]_{\mathtt{w}}$**.2**) we have

$$e'[\sigma[x := \nu]_{\mathtt{w}}[\![e_0]\!]]_{\mathtt{w}}[\nu := v[\![e_0]\!]]_{\mathtt{w}} \stackrel{\alpha}{\equiv} e'[\sigma[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}]_{\mathtt{w}}$$

Consequently we need only show that

$$\sigma[x := v]_{\mathtt{w}}[\![e_0]\!] \stackrel{\alpha}{\equiv} \sigma[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}$$

So choose $z \in \mathrm{Dom}(\sigma)$ arbitrarily. Then

$$\begin{aligned}
\sigma[x := v]_{\mathtt{w}}[\![e_0]\!](z) &= \sigma(z)[x := v]_{\mathtt{w}}[\![e_0]\!] \\
&\stackrel{\alpha}{\equiv} \sigma(z)[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}} && \text{by the induction hypothesis} \\
&= \sigma(z)[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}
\end{aligned}$$

(3. $e = \lambda z.e'$**)** In this case we have by computation

$$\begin{aligned}
e[x := v]_{\mathtt{w}}[\![e_0]\!] &= (\lambda z.e')[x := v]_{\mathtt{w}}[\![e_0]\!] \\
&\stackrel{\alpha}{\equiv} (\lambda \nu_0.e'[z := \nu_0]_{\mathtt{w}}[x := v]_{\mathtt{w}})[\![e_0]\!] && \text{for } \nu_0 \text{ suitably fresh} \\
&\stackrel{\alpha}{\equiv} \lambda \nu_0.(e'[z := \nu_0]_{\mathtt{w}}[x := v]_{\mathtt{w}}[\![e_0]\!]) && \text{since } \nu_0 \text{ is suitably fresh (lemma (.[\![.]\!]\textbf{.1}))} \\
&\stackrel{\alpha}{\equiv} \lambda \nu_0.(e'[z := \nu_0]_{\mathtt{w}}[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}) && \text{by the induction hypothesis and derived rule } (\stackrel{\alpha}{\equiv}\textbf{.6}) \\
&\stackrel{\alpha}{\equiv} (\lambda \nu_0.e'[z := \nu_0]_{\mathtt{w}}[x := \nu]_{\mathtt{w}}[\![e_0]\!])[\nu := v[\![e_0]\!]]_{\mathtt{w}} && \text{since } \nu_0 \text{ is suitably fresh} \\
&\stackrel{\alpha}{\equiv} (\lambda \nu_0.e'[z := \nu_0]_{\mathtt{w}}[x := \nu]_{\mathtt{w}})[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}} && \text{again since } \nu_0 \text{ is suitably fresh} \\
&\stackrel{\alpha}{\equiv} (\lambda \nu_0.e'[z := \nu_0]_{\mathtt{w}})[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}} && \text{again since } \nu_0 \text{ is suitably fresh} \\
&= e[x := \nu]_{\mathtt{w}}[\![e_0]\!][\nu := v[\![e_0]\!]]_{\mathtt{w}}
\end{aligned}$$

□