Section 0 : Introduction.

In this paper we investigate the first order metatheory of the classical propositional logic. In the first section we prove that the first order metatheory of the classical propositional logic is undecidable. Thus as a mathematical object even the simplest of logics is, from a logical standpoint, quite complex. In fact it is of the same complexity as true first order number theory.

This result answers negatively a question of J.F.A.K.Van Benthem (see [Van Benthem and Doets 1983]) as to whether the interpolation theorem in some sense completes the metatheory of the calculus. Let us begin by motivating the question that we answer. In [Van Benthem and Doets 1983] it is claimed that *a folklore predjudice has it that interpolation was the final elementary property of first order logic to be discovered.* Even though other properties of the propositional calculus have been discovered since Craig's original paper [Craig 1957] (see for example [Resnikoff 1965]) there is a lot of evidence for the fundamental nature of the property. In abstract model theory for example one finds that very few logics have the interpolation property. There are two well known open problems in this area. These are

1. *Is there a logic satisfying the full compactness theorem as well as the interpolation theorem that is not equivalent to first order logic even for finite models ?*

2. *Is there a logic stronger than L(Q), the logic with the quantifier* **there exists uncountably many**,*that is countably compact and has the interpolation property ?*

The first question is due to Friedman [Friedman 1975]; the basic question of whether or not compactness and interpolation characterize first order logic remains open. Mundici [Mundici 1981] has shown that if such a logic exists and has a weak Löwenheim-Skolem property then it will be the same as first order logic on countable models of finite type. The second and weaker question is attributed to Feferman [Makowsky, Shelah and Stavi 1976]. Here again there are only partial answers. Shelah recently claims to have shown that a positive answer to Feferman's question is consistent with ZFC. Perhaps it is to these sorts of results (or the lack of them) that folklore predjudice should look towards for its vindication. Further discussion of these topics can be found in [Barwise and Feferman], particularly in Kaufmann's contribution.

The idea behind Van Benthem's question was this: If we can show that the interpolation theorem plays an essential role in some formalized metatheory of the calculus then *folklore predjudice* has been vindicated. Now we already know that fragments of the metatheory of the propositional calculus are axiomatizable, notably the first order theories of <FORMULAS , BINARY CONSEQUENCE> and

<TYPES , INCLUSION>. The next reasonable level of structure seems to be

<FORMULAS , TYPES , BINARY CONSEQUENCE , INCLUSION , OCCURENCE>

whose theory includes the interpolation theorem. So the hope then is that the interpolation theorem will provide an essential link between the theories of these two substructures in order to axiomatize the whole.

The structure of this paper is as follows. In section one we prove that *folklore predjudice* must look elsewhere for its vindication, and in section two we sketch refinements of the method and their consequences. We wish to thank J.F.A.K.Van Benthem for mentioning the problem to us when he visited Stanford in 1982 and for comments on and corrections to an earlier version of this paper.

Section 1. The Classical Calculus

We begin with the necessary formalities .

Definition 1.1 : Let **i/** A $= \{p \leftarrow n\} \leftarrow n < \omega$ be a countable collection of propositional atoms. **ii/** P = The set of propositional formulas built up from A using $\neg$, $\wedge$ and $\rightarrow$. **iii/** L $= \{X | X \subset A, X$ finite $\}$. **iv/** O : P $\mapsto$ L is the function which takes a formula to the set of atoms which occur in it. **v/** $\vdash$ is the usual classical deduction relation on P, namely (x $\vdash$ y) iff (x $\rightarrow$ y) is a tautology. **vi/** $\subset$ is the subset relation on L.

Given this we can make the following definition:

Definition 1.2 :

1. PROP $= < P, L, \vdash, \subset, O >$.

2. Th(PROP) is the two-sorted** first order theory of PROP in the language of that structure.

It is Th(PROP) that we mean when we speak about the first order metatheory of the classical propositional calculus. Even though it is an impoverished version of what one might call the real first order metatheory of the logic (e.g one could include the semantics and operations corresponding to the connectives) , it suffices for our negative result.

Theorem 1.1 : Th(PROP) is hereditarily undecidable .

The separate structures $< P, \vdash>$ and $< L, \subset>$ both have decidable theories. The first is $\aleph \leftarrow 0$ categorical because after collapsing it under logical equivalence one obtains an atomless boolean algebra. The second being proved to be decidable by Skolem in what appears to be the very first use of the method of elimination of quantifiers (see for example [Skolem 1919] or [Feferman Vaught 1959]). As we began, it

** We use x,y,z,...to range over P and u,v,w,...to range over L was hoped that the interpolation theorem (which is written in the language of PROP below) would play a fundamental role in completeing the theory of the larger structure. In this sense our result that Th(PROP) has no complete recursive axiomatization is completely negative. Note that by Craig's trick not having a recursive axiomatization is equivalent to not having a recursively enumerable axiomatization. Hence propositional logic is axiomatizable, its metatheory is not.

Interpolation : Let $x \neq \top$ abbreviate the formula $\exists y(\neg(y \vdash x))$ and $x \neq \bot$ abbreviate the formula $\exists y(\neg(x \vdash y))$. Then the Interpolation Theorem is

$$\forall x, y((x \vdash y \wedge x \neq \bot \wedge y \neq \top) \rightarrow \exists z(x \vdash z \wedge z \vdash y \wedge O(z) \subset O(x) \cap O(y)))$$

The proof of Theorem 1.1 is essentially straightforward and consists in showing that there is an effective procedure whereby given any sentence Φ in the language of a single binary relation, R , one can find a sentence Φ^{PROP} in the language of PROP such that the following are equivalent.

1. Φ is valid in all finite models that interpret R as an irreflexive relation.

2. PROP $\models \Phi^{PROP}$.

The former question is known to be undecidable ; see for example [Rabin 1965]. This equivalence is obtained by showing how one can represent finite irreflexive τ-structures ($\tau = \{\mathrm{R}\}$) and express first order properties of such structures within PROP. To be explicit, a generic τ-structure $\mathfrak{A} =< A, R >$ will be coded up by two elements x and y of P such that x will uniquely determine a domain A^x, and relative to that x, y will uniquely determine an irreflexive relation $R^x \leftarrow y$ on A^x. Furthermore given any finite irreflexive τ-structure $\mathfrak{A}$, we can find $x, y \in$ P such that $\mathfrak{A} \cong \mathfrak{A}^x \leftarrow y$ where $\mathfrak{A}^x \leftarrow y =< A^x, R^x \leftarrow y >$.

Without further ado we set about proving the theorem .

Proof of Theorem 1.1 : We can simplify matters without loss of generality by dealing with the structure we obtain from PROP by collapsing P to the set of equivalence classes of the relation

$$\{< x, y > | x, y \in P, x \vdash y, y \vdash x \wedge O(x) = O(y)\}.$$

We shall continue to call the structure thus obtained PROP and its altered domain P. An element z of P will be called trivial if it can be deduced from anything or if everything can be deduced from it (i.e $z = \top \vee z = \bot$). Given any propositional atom , p , there are exactly two nontrivial elements of P , x and y , such that O(x) = O(y) = $\{p\}$; these correspond to the, equivalence classes of the, formulas p and ¬p . We call these the atomic formulas in $\{p\}$ and denote them by a(p) . For x $\in$ P we say x $\in \Gamma$ if there is some propositional atom p such that x $\in$ a(p). By a further abuse of notation we shall take x $\in \Gamma \cap O$(z) to mean that x $\in \Gamma$ and $O(\mathrm{x}) \subset O(\mathrm{z})$.

To begin with we must define some predicates . They will all be expressible in the language of PROP with the appropriate free variables . A word about notation might be helpful. Any notion that appears as $[\![\textbf{notion}]\!]$ will denote a formula in the language of PROP (whose meaning is taken to be that notion or its representation within the strucutre PROP). In general however we will not explicitly write out the corresponding formula but be content with giving a description of it.

1. $[\![\textbf{Choice}]\!]$x denotes the formula expressing that x $\in$ P , and for each $\{p\} \subset O(x)$ exactly one of a(p), y, is such that y $\vdash$ x . Call this element p^x and the other $\neg p^x$. To be explicit , $[\![\textbf{Choice}]\!]x$ is the formula

$\forall v((v \subset O(x) \wedge [\![\textbf{Singleton}]\!]v) \rightarrow \exists! y(O(y) = v \wedge y \vdash x \wedge y \neq \top \wedge y \neq \bot))$

where $[\![\textbf{Singleton}]\!]v$ is $\forall u(u \subset v \rightarrow (u = v \vee \forall w(u \subset w)))$.

2. $[\![\textbf{Domain}]\!]$x holds iff $[\![\textbf{Choice}]\!]$x and for any y $\in$ P such that if **i/** O(x) = O(y), **ii/** $[\![\textbf{Choice}]\!]$y and **iii/** $p^x = p^y$ for $\{p\} \subset O(x)$, then x $\vdash$ y.

It is easily seen that if $[\![\textbf{Domain}]\!]$x then x $= \bigvee\{p^x : \{p\} \subset \mathrm{O(x)}\}$. The reason behind the definition of $[\![\textbf{Domain}]\!]$x is that it enables us to distinguish between the two elements of a(p) in a predetermined manner, as long as p is a member of $O(\mathrm{x})$. This is easily seen to be necessary since no formula of PROP can, without parameters, distinguish between a propositional atom and its negation.

Now given any x such that $[\![\textbf{Domain}]\!]$x, we define what it means for y $\in$ P to define a binary irreflexive relation on x.

3. $[\![\textbf{y is a relation on x}]\!]$ holds iff the following two conditions are satisfied:

i/ $\forall p \leftarrow i, q \leftarrow i \in P$ (i = 1,2) such that $p \leftarrow i, q \leftarrow i \in \Gamma \cap O(\mathrm{x})$ are distinct, i.e. $O(p \leftarrow i) \cap O(q \leftarrow i) = \emptyset$, if $\exists \alpha \in \Gamma$, $\alpha \notin \Gamma \cap O(\mathrm{x})$ with $(p \leftarrow i^x \wedge \neg q \leftarrow i^x \wedge \alpha) \vdash$ y (i = 1,2) then $p \leftarrow 1^x = p \leftarrow 2^x$ and $q \leftarrow 1^x = q \leftarrow 2^x$.

ii/ $\forall$ p,q $\in$ P such that p,q $\in \Gamma \cap O(\mathrm{x})$ are distinct, if $\exists \alpha \leftarrow i \in \Gamma$ (i = 1,2) $\alpha \leftarrow i \notin \Gamma \cap O(\mathrm{x})$ such that $(p^x \wedge \neg q^x \wedge \alpha \leftarrow i) \vdash$ y (i = 1,2) then $\alpha \leftarrow 1 = \alpha \leftarrow 2$.

Given that y satisfies $[\![\textbf{y is a relation on x}]\!]$ and $[\![\textbf{Domain}]\!]\textbf{x}$ holds, we can say what relation y is supposed to code up.

4. $R^x \leftarrow y\{p\},\{q\}$ is defined to mean p,q $\in \Gamma \cap O(\mathrm{x})$ are distinct, i.e. $p^x \neq q^x$, and $\exists \alpha \in \Gamma$, $\alpha \notin \Gamma \cap O(\mathrm{x})$, such that $(p^x \wedge \neg q^x \wedge \alpha) \vdash$ y.

Now we are in a position to explain our coding.

We abbreviate the formula $[\![\textbf{Domain}]\!]\mathbf{x} \wedge [\![\textbf{y is a relation on x}]\!]$ by the expression $\mathfrak{A}[\![x, y]\!]$.

If $[\![\textbf{Domain}]\!]\mathbf{x}$ holds then we let $A^x = \{v : [\![\textbf{Singleton}]\!]v \wedge v \subset O(x)\}$ and $\mathfrak{A}^x \leftarrow y =< A^x, R^x \leftarrow y >$.

All this leads to the following:

Lemma: Suppose $\mathfrak{A} =< A, R >$ where A is finite and R is a binary irreflexive relation on A. Then $\exists x, y \in$ P such that 1. $\mathfrak{A}[\![x, y]\!]$ and 2. $\mathfrak{A} \cong \mathfrak{A}^x \leftarrow y$.

Proof of Lemma: Suppose A = $\{0, 1, ..., n\}$ and |R| = k. Enumerate R without repetitions:

R = $\{r \leftarrow 1, r \leftarrow 2, .., r \leftarrow k\}$. Let x = $(p \leftarrow 0 \vee p \leftarrow 1 \vee ... \vee p \leftarrow n)$ and y = $\bigvee\{(p \leftarrow i \wedge \neg p \leftarrow j \wedge p \leftarrow n + s) : s = 1, .., k \wedge r \leftarrow s =< i, j >\}$.

Thus we must prove that these have the desired properties. Clearly $[\![\textbf{Domain}]\!]\mathbf{x}$ holds. We show that $[\![\textbf{y is a relation on x}]\!]$ and that the map Υ:$\{p \leftarrow i\} \mapsto i$ (i = 0,..,n) defines an isomorphism $\mathfrak{A}^x \leftarrow y \cong \mathfrak{A}$.

To prove $[\![\textbf{y is a relation on x}]\!]$ we must prove two things, 3i/ and 3ii/. We shall prove 3i/; 3ii/ follows by similar reasoning. Suppose p,q $\in \Gamma \cap O$(x) are distinct, $(p^x \neq q^x)$, and $\alpha \in \Gamma$, $\alpha \notin \Gamma \cap O$(x) are such that $(p^x \wedge \neg q^x \wedge \alpha) \vdash$ y. We show firstly that $\alpha = p \leftarrow n + s$ for some s = 1,..,k and secondly that $p^x = p \leftarrow i$ and $q^x = p \leftarrow j$ where $r \leftarrow s =< i, j >$.

Suppose $\alpha \neq p \leftarrow n + s$ for s = 1,..,k; then either $\alpha = \neg p \leftarrow n + s$ for some s = 1,..,k or $O(\alpha) \subset \{p \leftarrow e\} \leftarrow e > n + k$. In either case one can find an assignment v : $\{p \leftarrow i\} \leftarrow i \in \omega \mapsto \{0, 1\}$ such that v $\models (p^x \wedge \neg q^x \wedge \alpha)$ but $\neg$ (v $\models y$), a contradiction. Therefore $\alpha = p \leftarrow n + s$ for some s = 1,..,k.

Now to show that $p^x = p \leftarrow i^x$ and $q^x = q \leftarrow j^x$ (where $r \leftarrow s =< i, j >$), observe that since O(p),O(q) $\subset \{p \leftarrow 0, p \leftarrow 1, .., p \leftarrow n\}$, a simple semantic argument like the above will show that $(p^x \wedge \neg q^x \wedge p \leftarrow n + s) \models (p \leftarrow i \wedge \neg p \leftarrow j \wedge p \leftarrow n + s)$.

By definition of x (and by assumption on p and q), we have that for some a,b $\in \{0, 1, .., n\}, (a \neq b)$, $p^x = p \leftarrow a$ and $q^x = p \leftarrow b$. Hence $(p \leftarrow a \wedge \neg p \leftarrow b \wedge p \leftarrow n + s) \models (p \leftarrow i \wedge \neg p \leftarrow j \wedge p \leftarrow n + s)$. Thus we conclude that a = i and b = j.

To show that Υ is an isomorphism it suffices to show that $R^x \leftarrow y\{p \leftarrow i\}, \{p \leftarrow j\} \Longleftrightarrow$ Ri,j , since Υ is clearly a bijection.

$(\Longrightarrow)$ Suppose $R^x \leftarrow y\{p \leftarrow i\}, \{p \leftarrow j\}$ holds; then $\exists \alpha \in \Gamma, \alpha \notin \Gamma \cap O(x)$ such that $(p \leftarrow i^x \wedge \neg p \leftarrow j^x \wedge \alpha) \vdash y$. In other words $(p \leftarrow i \wedge \neg p \leftarrow j \wedge \alpha) \vdash y$. But by definition this implies $\alpha = p \leftarrow n + s$ and $r \leftarrow s =< i, j >$. Thus Ri,j.

$(\Longleftarrow)$ Suppose Ri,j holds; then $(p \leftarrow i \wedge p \leftarrow j \wedge p \leftarrow n + s) \vdash y$ where $r \leftarrow s =< i, j >$. Thus

$R^x \leftarrow y\{p \leftarrow i\}, \{p \leftarrow j\}$

Q.E.D.

We can now set about defining our procedure for finding Φ^{PROP} from Φ . The first step is to define

4. $[\![\mathfrak{A}^x \leftarrow y \models \Phi]\!]$

which holds iff $\mathfrak{A}^x \leftarrow y \models \Phi$ (assuming $\mathfrak{A}[\![x, y]\!]$ holds).

This is done by induction on formulae in the language of R:

i/ $\Phi = \mathrm{Ruv} \Longleftrightarrow [\![\mathfrak{A}^x \leftarrow y \models \Phi]\!] = R^x \leftarrow y\mathrm{u,v}$

ii/ $\Phi = (\mathrm{u} = \mathrm{v}) \Longleftrightarrow [\![\mathfrak{A}^x \leftarrow y \models \Phi]\!] = (\mathrm{u} = \mathrm{v})$

iii/ $\Phi = \neg\Psi \Longleftrightarrow [\![\mathfrak{A}^x \leftarrow y \models \Phi]\!] = \neg[\![\mathfrak{A}^x \leftarrow y \models \Psi]\!]$

iv/ $\Phi = \Theta \leftarrow 0 \wedge \Theta \leftarrow 1 \Longleftrightarrow [\![\mathfrak{A}^x \leftarrow y \models \Phi]\!] = [\![\mathfrak{A}^x \leftarrow y \models \Theta \leftarrow 0]\!] \wedge [\![\mathfrak{A}^x \leftarrow y \models \Theta \leftarrow 1]\!]$

v/ $\Phi = \forall v\ \Psi \Longleftrightarrow [\![\mathfrak{A}^x \leftarrow y \models \Phi]\!] = \forall v\ ((v \in A^x) \rightarrow [\![\mathfrak{A}^x \leftarrow y \models \Psi]\!]\ )$.

Now given a sentence Φ in the language of R we define

5. $\Phi^{PROP} = \forall\ \mathrm{x,y}\ (\mathfrak{A}[\![\mathrm{x,y}]\!] \rightarrow [\![\mathfrak{A}^x \leftarrow y \models \Phi]\!])$.

It is now easy to check that this has the desired properties , and so the theorem is proved and section one completed . **Q.E.D**

Section 2. Refinements of the method

In the first section we showed how one could represent and hence quantify over finite irreflexive τ-structures within PROP. Furthermore for any first order sentence in the language of τ we could express the fact that a τ-structure, $\mathfrak{A}$, satisfied that sentence. In this section we shall show how this can be extended so that one can express monadic second order properties of such structures and certain algebraic relations between such structures. This has the following two consequences:

Theorem 2.1 : Th(PROP) is $\Delta \leftarrow 1^1$.

This is proven by showing that true first order arithmetic is interpretable within PROP. Actually it is easy to see that they are equivalent since PROP is interpretable within arithmetic. Theorem 2.1 was pointed out to me by J.F.A.K.Van Benthem in his comments and corrections of an earlier draft.

The second result concerns so called infinitary analogues of PROP which were anounced in [Mason 84]. Define (for each infinite cardinal κ) $PROP \leftarrow \kappa$ to be the following:

Definition 2.1 : Let $PROP \leftarrow \kappa = < P \leftarrow \kappa, L \leftarrow \kappa, \vdash, \subset, O>$, where $A \leftarrow \kappa = \{p \leftarrow \alpha\} \leftarrow \alpha < \kappa^*$ is a set of propositional atoms, and κ^* is the smallest regular cardinal greater than or equal to κ . $P \leftarrow \kappa$ is the set of propositional formula built up from $A \leftarrow \kappa$ using $\neg$, $\wedge$, $\rightarrow$ and $\bigwedge$ (the last being the conjunction of less than κ formula) . $\vdash$, O and $\subset$ are the entirely analogous counterparts to those defined in section 1 , and $L \leftarrow \kappa$ is the image of $P \leftarrow \kappa$ under the mapping O.

Theorem 2.2 : i/ If $\kappa \leftarrow 0 = \aleph \leftarrow \alpha$, $\alpha < \omega^{\omega}$, and $\kappa \leftarrow 0 \neq \kappa \leftarrow 1$ then $PROP \leftarrow \kappa \leftarrow 0 \not\equiv PROP \leftarrow \kappa \leftarrow 1$

ii/ If $< \kappa \leftarrow 0, \in > \not\equiv \leftarrow L < \kappa \leftarrow 1, \in >$ where L is monadic second order logic augmented with quantification over unary functions , then $PROP \leftarrow \kappa \leftarrow 0^{+} \not\equiv PROP \leftarrow \kappa \leftarrow 1^{+}$.

Concerning the relation between PROP and $PROP \leftarrow \kappa$ we remark that there is a slightly stronger version of interpolation that out of all the $PROP \leftarrow \kappa$, only PROP (i.e $PROP \leftarrow \aleph \leftarrow 0$) satisfies . So in this sense there is a grain of truth to the claim that the stronger interpolation theorem in some sense completes the theory of PROP . We should also add that the question of which $PROP \leftarrow \kappa$ actually have the weaker interpolation property is a difficult one (see [Friedman 1975]).

Type minimal Interpolation : Let $INT(x,y,z)$ abbreviate the formula $(x \vdash z \wedge z \vdash y \wedge O(z) \subset O(x) \cap O(y))$. Then the stronger version of Interpolation is just

$\forall x,y((x \vdash y \wedge x \neq \bot \wedge y \neq \top) \rightarrow \exists z \leftarrow 0(INT(x,y,z \leftarrow 0) \wedge \forall z \leftarrow 1(INT(x,y,z \leftarrow 1) \rightarrow \neg(O(z \leftarrow 1) \subset O(z \leftarrow 0) \wedge O(z \leftarrow 1) \neq O(z \leftarrow 0)))))$.

We shall only hint at the proofs of these theorems, it being sufficient to sketch the general method. Let us begin by showing how we can express monadic second order properties of a particular $\mathfrak{A}^{x} \leftarrow y$. Suppose $\mathfrak{A}[\![x,y]\!]$ holds, then we define:

6. z[[**Subset**]]x holds iff **i/** [[**Domain**]]z **ii/** O(z) $\subset$ O(x) **iii/** z $\vdash$ x.

We write $A^{z} \subset A^{x}$ when z[[**Subset**]]x holds. Notice that by the definition we have $\mathfrak{A}^{z} \leftarrow y$ is a substructure of $\mathfrak{A}^{x} \leftarrow y$. Furthermore if $x \leftarrow 1$ and $x \leftarrow 2$ are domains that are compatable on their intersection, i.e $\forall\{p\} \in (A^{x\leftarrow 1} \cap A^{x\leftarrow 2})(p^{x\leftarrow 1} = p^{x\leftarrow 2})$, then we can easily define $(A^{x\leftarrow 1} \cup A^{x\leftarrow 2})$ to be A^{x} where x = $(x \leftarrow 1 \vee x \leftarrow 2)$, and similarly with the other set theoretic operations. Thus if we add the following two clauses to the definition of $[\![\mathfrak{A}^{x} \leftarrow y \models \Phi]\!]$, (4.), where Φ is now a monadic second order formula written in the language τ, we obtain a definition of monadic second order satisfaction within PROP.

vi/ $\Phi = (v \in Z) \Longleftrightarrow [\![\mathfrak{A}^{x} \leftarrow y \models \Phi]\!] = v \in A^{z}$.

vii/ $\Phi = \forall Z\Psi \Longleftrightarrow [\![\mathfrak{A}^{x} \leftarrow y \models \Phi]\!] = \forall z((A^{z} \subset A^{x}) \rightarrow [\![\mathfrak{A}^{x} \leftarrow y \models \Psi]\!])$.

Even though the distinction between first order and monadic second order logic collapses for finite τ-

structures our expressive abilities are improved substantially when we are working within $PROP \leftarrow \kappa$. For example, we can now quantify over well orderings.

So far we have shown how to express monadic second order properties. Now is the time to describe how we can represent and thus quantify over unary functions. Suppose $x \leftarrow 1$ and $x \leftarrow 2$ are two arbitrary domains. We wish to be able to represent a function $F : x \leftarrow 1 \mapsto x \leftarrow 2$. This is done as follows:

7. A function F is a triple $< x, y \leftarrow 1, y \leftarrow 2 >$ of elements from P such that

i/ $[\![$**Domain**$]\!]$**x** and $A^x \cap A^{x\leftarrow 1} = A^x \cap A^{x\leftarrow 2} = \emptyset$.

ii/ $[\![y \leftarrow i$ **is a relation on** $(x \vee x \leftarrow i)]\!]$, i=1,2.

iii/ $R^{(x\leftarrow 1 \vee x)} \leftarrow y \leftarrow 1$ is a function from $A^{x\leftarrow 1}$ to A^x.

iv/ $R^{(x\leftarrow 2 \vee x)} \leftarrow y \leftarrow 2$ is a function from A^x to $A^{x\leftarrow 2}$.

To express that a triple $< x, y \leftarrow 1, y \leftarrow 2 >$ satisfies the above we write $F[\![x, y \leftarrow 1, y \leftarrow 2]\!]$. We also write $F^x \leftarrow y \leftarrow 1y \leftarrow 2$ for the corresponding function, which is defined by:

8. $F^x \leftarrow y \leftarrow 1y \leftarrow 2(u) = v$ iff $u \in A^{x\leftarrow 1}$ and $v \in A^{x\leftarrow 2}$ and $\exists w \in A^x$ such that $R^{(x\leftarrow 1 \vee x)} \leftarrow y \leftarrow 1u, w$ and $R^{(x\leftarrow 2 \vee x)} \leftarrow y \leftarrow 2w, v$.

Thus by quantifying over triples we can quantify over unary functions. For example we can add the following two clauses to obtain a definition of satisfaction for the logic L mentioned in Theorem 2.2:

viii/ $\Phi = (F(u) = v) \Longleftrightarrow [\![\mathfrak{A}^x \leftarrow y \models \Phi]\!] = F^z \leftarrow y \leftarrow 1y \leftarrow 2(u) = v$.

ix/ $\Phi = \forall F \Psi \Longleftrightarrow [\![\mathfrak{A}^x \leftarrow y \models \Phi]\!] = \forall z, y \leftarrow 1, y \leftarrow 2((F[\![z, y \leftarrow 1, y \leftarrow 2]\!] \wedge F^z \leftarrow y \leftarrow 1y \leftarrow 2 : A^x \mapsto A^x) \longrightarrow \mathfrak{A}^x \leftarrow y \models \Psi]\!])$.

The proofs of both theorems are now quite straightforward. One specializes to the case where the τ-structures one is considering are linear orders. In the first case one identifies numbers with linear orderings. Using the above machinary, it is quite straightforward to express that one linear order is the sum or product of two other linear orders. The second theorem is just as straightforward. One shows that one can quantify over cardinals and ordinals. Theorem 2.2 i/, for example, follows from the observation that if $\alpha < \omega^\omega$; then there are n , $m \leftarrow 0, ..., m \leftarrow n - 1 \in \omega$ such that

$\alpha = m \leftarrow n - 1.\omega^{n-1} + ... + m \leftarrow 0.\omega^0$. And this can easily be expressed in by sentence.

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