Propositional Logic of Context

Saša Buvač & Ian A. Mason
Computer Science Department
Stanford University
Stanford
California 94305-2140
{buvac, iam}@sail.stanford.edu

Abstract

In this paper we investigate the simple logical properties of *contexts*. We describe both the syntax and semantics of a general propositional language of context, and give a Hilbert style proof system for this language. A propositional logic of context extends classical propositional logic in two ways. Firstly, a new modality, $\texttt{ist}(\kappa, \phi)$, is introduced. It is used to express that the sentence, ϕ, holds in the context κ. Secondly, each context has its own vocabulary, i.e. a set of propositional atoms which are *defined* or *meaningful* in that context. The main results of this paper are the soundness and completeness of this Hilbert style proof system. We also provide soundness and completeness results (i.e. correspondence theory) for various extensions of the general system.

Introduction

In this paper we investigate the simple logical properties of *contexts*. Contexts were first introduced into AI by John McCarthy in his Turing Award Lecture, [McCarthy, 1987], as an approach which might lead to the solution of the problem of *generality* in AI. This problem is simply that existing AI systems *lack* generality.

Since then, contexts have found a large number of uses in various areas of AI. R. V. Guha's doctoral dissertation [Guha, 1991] under McCarthy's supervision was the first in-depth study of context. Guha's context research was primarily motivated by the Cyc system [Guha and Lenat, 1990] (a large common-sense knowledge-base currently being developed at MCC). Without using contexts it would have been virtually impossible to create and successfully use a knowledge base of the size of Cyc.

Large knowledge bases are not the only place where contexts have found practical use. The knowledge sharing community has accepted the need for explicating context when transferring information from one agent to another. Currently, proposals for introducing contexts into the Knowledge Interchange Format or KIF [Genesereth and Fikes, 1992] are being considered. Furthermore, it seems that the context formalism can provide semantics for the process of translating facts into KIF and from KIF, one of the key tasks that the knowledge sharing effort is facing.

The meaning of an utterance depends on the context in which it is uttered. Computational linguists have developed various ways of describing this context. For example, Barbara Grosz in her Ph.D. thesis, [Grosz, 1977], implicitly captures the context of a discourse by *focusing* on the objects and actions which are most relevant to the discourse. This representation is similar to an ATMS context [de Kleer, 1986], which is simply a list of propositions that are *assumed* by the reasoning system.

However till now no formal logical explication of contexts has been given. The aim of this paper is to rectify this deficiency. We describe both the syntax and semantics of a general propositional language of context, and give a Hilbert style proof system for this language. The main results of this paper are the soundness and completeness of this Hilbert style proof system. We also provide soundness and completeness results (i.e. correspondence theory) for various extensions of the general system.

Notation

We use standard mathematical notation. If X and Y are sets, then $X \to_{\mathrm{p}} Y$ is the set of partial functions from X to Y. $\mathbf{P}(X)$ is the set of subsets of X. X^* is the set of all finite sequences, and we let $\bar{x} = [x_1, \ldots, x_n]$ range over X^*. ϵ is the empty sequence. We use the infix operator $*$ for appending sequences. We make no distinction between an element and the singleton sequence containing that element. Thus we write $\bar{x} * x_1$ instead of $\bar{x} * [x_1]$. As is usual in logic we treat X^* as a tree (that grows downward). $\bar{x}_1 < \bar{x}_0 \leq \epsilon$ iff $\bar{x}_1$ properly extends $\bar{x}_0$ (i.e. $(\exists \bar{y} \in X^* - \{\epsilon\})(\bar{x}_1 = \bar{x}_0 * \bar{y})$). We say $Y \subseteq X^*$ is a subtree rooted at $\bar{y}$ to mean

1. $\bar{y} \in Y$ and $(\forall \bar{z} \in Y)(\bar{z} \leq \bar{y})$
2. $(\forall \bar{z} \in Y)(\forall \bar{w} \in X^*)(\bar{z} \leq \bar{w} \leq \bar{y} \to \bar{w} \in Y)$

The General System

A propositional logic of context extends classical propositional logic in two ways. Firstly, a new modality, $\mathtt{ist}(\kappa, \phi)$, is introduced. It is used to express that the sentence, ϕ, holds in the context κ. Secondly, each context has its own vocabulary, i.e. a set of propositional atoms which are *defined* or *meaningful* in that context. The vocabulary of one context may or may not overlap with another context.

Syntax

We begin with two distinct countably infinite sets, $\mathbb{K}$ the set of all contexts, and $\mathbb{P}$ the set of propositional atoms. The set, $\mathbb{W}$, of well-formed formulas (wffs) is built up from the propositional atoms, $\mathbb{P}$, using the usual propositional connectives (negation and implication) together with the **ist** modality.

Definition ($\mathbb{W}$):

$$\mathbb{W} = \mathbb{P} \cup (\neg\mathbb{W}) \cup (\mathbb{W} \rightarrow \mathbb{W}) \cup \mathtt{ist}(\mathbb{K}, \mathbb{W})$$

The operations $\wedge$, $\vee$ and $\leftrightarrow$ are defined as abbreviations in the usual way. The term *literal* is used to refer to a propositional atom or the negation of a propositional atom. We use $\pm\phi$ to represent either the formula ϕ, or its negation $\neg\phi$. We also use the following abbreviations:

$$\mathtt{ist}(\overline{\kappa}, \phi) \quad := \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \ldots, \mathtt{ist}(\kappa_n, \phi)))$$
$$\mathtt{ist}^{\pm}(\overline{\kappa}, \phi) \quad := \pm\mathtt{ist}(\kappa_1, \pm\mathtt{ist}(\kappa_2, \cdots \pm \mathtt{ist}(\kappa_n, \phi) \cdots))$$

when $\overline{\kappa}$ is the context sequence $[\kappa_1, \kappa_2, \ldots, \kappa_n]$. In the definition of $\mathtt{ist}^{\pm}$ all the **ist**'s need not be of the same parity. PROP is the set of all well formed formulas which do not contain **ist**'s. If ψ is a formula containing distinct atoms $p_1, \ldots, p_n$, then we write $\psi(\phi_1, \ldots, \phi_n)$ for the formula which results from ψ by simultaneously replacing all the occurrences of p_i in ψ by ϕ_i. We say that $\psi(\phi_1, \ldots, \phi_n)$ is an *instance* of ψ.

Semantics

We begin with a system which makes as few semantic restrictions as possible. Other systems are obtained by placing restrictions on the models. The semantics of the general system has the following three features:

Firstly, the nature of a particular context may itself be context dependent. For example, in the context of the 1950's, the context of car racing is different than than the context of car racing viewed from today's context. This leads naturally to considering sequences of contexts rather than a solitary context. We refer to this feature of the system as *non-flatness*. It reflects on the intuition that what holds in a context can depend on how this context has been reached, i.e. from which perspective it is being viewed. For example, non-flatness will be desirable if we represent the beliefs of an agent as the sentences which hold in a context. A system of flat contexts can easily be obtained by placing certain restrictions on what kinds of structures are allowed as models, as well as enriching the axiom system.

Secondly, a context is modelled by a set of truth assignments, that describe the possible states of affairs of that context. Therefore the **ist** modality is interpreted as validity: $\mathtt{ist}(\kappa, \rho)$ is true iff the propositional atom ρ is true in all the truth assignments associated with context κ. Treatment of **ist** as validity corresponds to Guha's proposal for context semantics, which was motivated by the Cyc knowledge base. A system which models a context by a single truth assignment, thus interprets **ist** as truth, can be obtained by placing simple restrictions on the definition of a model, and enriching the set of axioms.

Thirdly, since different contexts can have different vocabularies, some propositions can be meaningless in some contexts, and therefore the truth assignments describing the state of affairs in that context need to be partial.

Definition ($\mathfrak{M}$): In this system a model, $\mathfrak{M}$, will be a function which maps a context sequence $\overline{\kappa} \in \mathbb{K}^*$ to a set of partial truth assignments,

$$\mathfrak{M} \in \mathbb{K}^* \rightarrow_{\mathrm{p}} \mathbf{P}(\mathbb{P} \rightarrow_{\mathrm{p}} 2),$$

with the added conditions that

1. $(\forall\overline{\kappa})(\forall\nu_1, \nu_2 \in \mathfrak{M}(\overline{\kappa}))(\mathrm{Dom}(\nu_1) = \mathrm{Dom}(\nu_2))$
2. $\mathrm{Dom}(\mathfrak{M})$ is a subtree of $\mathbb{K}^*$ rooted at some context sequence $\overline{\kappa}_0$.

We write $\overline{\kappa}^{\mathfrak{M}}$ to denote the set of partial truth assignments $\mathfrak{M}(\overline{\kappa})$. Note that $\overline{\kappa}^{\mathfrak{M}}$ can be empty. The collection of all such models will be denoted by $\mathbb{M}$.

We could have assumed the existence of a *fixed outermost context* which would result in $\mathrm{Dom}(\mathfrak{M})$ being a tree rooted at empty sequence ϵ (i.e. the fixed outermost context). This would result in slightly simpler notation and proofs. However, although more complicated, our definition is based on the intuition that there is no *outermost* context.

Vocabularies The truth assignments in our model are partial. The atoms which are given a truth value in a context are defined by a relation $\mathsf{Vocab} \subseteq \mathbb{K}^* \times \mathbb{P}$.

Definition (*Vocab* of $\mathfrak{M}$): We define a function $\mathit{Vocab} : \mathbb{M} \rightarrow \mathbf{P}(\mathbb{K}^* \times \mathbb{P})$ which given a model returns the vocabulary of the model:

$$\mathit{Vocab}(\mathfrak{M}) := \{<\overline{\kappa}, \rho> \mid \overline{\kappa} \in \mathrm{Dom}(\mathfrak{M}) \text{ and } \rho \in \mathrm{Dom}(\mathfrak{M}(\overline{\kappa}))\}$$

We say that a model $\mathfrak{M}$ is *classical on vocabulary* Vocab iff $\mathsf{Vocab} \subseteq \mathit{Vocab}(\mathfrak{M})$.

The notion of vocabulary can also be applied to sentences. Intuitively, the vocabulary of a sentence relates a context sequence to the atoms which occur in the scope of that context sequence. In the definition we also need to take into account that sentences are not given in isolation but in a context.

Definition (Vocab **of** ϕ **in** $\overline{\kappa}$): We define a function Vocab : $\mathbb{K}^* \times \mathbb{W} \rightarrow \mathbf{P}(\mathbb{K}^* \times \mathbb{P})$ which given formula in a context, returns the vocabulary of the formula. Vocab$(\overline{\kappa}, \phi)$ is defined inductively by:

$\{<\overline{\kappa}, \phi>\}$	$\phi \in \mathbb{P}$
Vocab$(\overline{\kappa}, \phi_0)$	$\phi = \neg\phi_0$
Vocab$(\overline{\kappa} * \kappa, \phi_0)$	$\phi = \mathtt{ist}(\kappa, \phi_0)$
Vocab$(\overline{\kappa}, \phi_0) \cup$ Vocab$(\overline{\kappa}, \phi_1)$	$\phi = \phi_0 \rightarrow \phi_1$

It is extended to sets of formulas in the obvious way.

Note that it is only in the propositional case that we can carry out this *static* analysis of the vocabulary of a sentence. It will not be possible in the quantified versions. Also note that our definition of vocabulary of a sentence is somewhat different from Guha's notion of definedness. Guha proposes to treat $\mathtt{ist}(\kappa, \phi)$ as false if ϕ is not in the vocabulary of the context κ.

Satisfaction We can think of partial truth assignments as total truth assignments in a three-valued logic. Our satisfaction relation then corresponds to Bochvar's three valued logic [Bochvar, 1972], since an implication is meaningless if either the antecedent or the consequent are meaningless. We chose Bochvar's three valued logic because we intend meaningfulness to be interpreted as syntactic meaningfulness, rather than semantic meaningfulness along the lines of Kleene's three valued logic [Kleene, 1952].

Definition ($\models$):
If $\nu \in \overline{\kappa}^{\mathfrak{M}}$ and Vocab$(\overline{\kappa}, \varphi) \subseteq$ *Vocab*$(\mathfrak{M})$, then

$\mathfrak{M}, \nu \models_{\overline{\kappa}} \rho$ iff $\nu(\rho) = 1, \quad \rho \in \mathbb{P}$

$\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\phi$ iff not $\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi$

$\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi \rightarrow \psi$ iff $\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi$ implies $\mathfrak{M}, \nu \models_{\overline{\kappa}} \psi$

$\mathfrak{M}, \nu \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi)$ iff $\forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}}\ \mathfrak{M}, \nu_1 \models_{\overline{\kappa} * \kappa_1} \phi$

In the last point note that $\overline{\kappa} * \kappa_1 \in$ Dom$(\mathfrak{M})$ since the Dom$(\mathfrak{M})$ is a rooted subtree, and Vocab$(\overline{\kappa}, \phi) \subseteq$ *Vocab*$(\mathfrak{M})$.

We write $\mathfrak{M} \models_{\overline{\kappa}} \phi$ iff $\forall \nu \in \overline{\kappa}^{\mathfrak{M}} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \phi$.

Formal System

We now present the formal system. To do this we fix a particular vocabulary, Vocab $\subset \mathbb{K}^* \times \mathbb{P}$, and define a provability relation, $\vdash_{\overline{\kappa}}^{\mathsf{Vocab}}$. Since Vocab will remain fixed throughout we omit explicitly mentioning it and write $\vdash_{\overline{\kappa}} \phi$ instead. Similarly, to avoid constantly stating lengthy side conditions we make the following convention.

Definedness Convention: *In the sequel, whenever we write $\vdash_{\overline{\kappa}} \phi$ we will be assuming implicitly that* Vocab$(\overline{\kappa}, \phi) \subseteq$ Vocab.

Axioms and inference rules are given in table 1. Note that the rules of inference preserve the (**definedness convention**).

Assuming that our system was limited to only one context, the rule (**CS**) would be identical to the rule of necessitation in normal systems of modal logic, and axiom schema (**K**) would be identical to the the standard axiom schema K. Thus in the single context case, ignoring axiom schemas $(\triangle_+)$ and $(\triangle_-)$, our formal system is identical to what is usually called the *normal system* of modal logic, characterized by (**PL**), (**MP**), (**K**), and the rule of necessitation. The axiom schemas $(\triangle_+)$ and $(\triangle_-)$ are needed in order to accommodate the validity aspect of the ist modality. It turns that they derivable in the system which treats ist as truth and does not allow inconsistent contexts.

Provability A formula ϕ *is provable in context $\overline{\kappa}$ with vocabulary* Vocab (formally $\vdash_{\overline{\kappa}} \phi$) iff $\vdash_{\overline{\kappa}} \phi$ is an instance of an axiom schema or follows from provable formulas by one of the inference rules; formally, iff there is a sequence $[\vdash_{\overline{\kappa}_1} \phi_1, \ldots, \vdash_{\overline{\kappa}_n} \phi_n]$ such that $\overline{\kappa}_n = \overline{\kappa}$, and $\phi_n = \phi$ and for each $i \leq n$ either $\vdash_{\overline{\kappa}_i} \phi_i$ is an axiom, or is derivable from the earlier elements of the sequence via one of the inference rules. In the case of assumptions, formula ϕ is provable from assumptions T in context $\overline{\kappa}_0$ with vocabulary Vocab (formally T$\vdash_{\overline{\kappa}_0}^{\mathsf{Vocab}}$, or again taking into account that Vocab is fixed T $\vdash_{\overline{\kappa}_0} \phi$) iff there are formulas $\phi_1, \ldots, \phi_n \in$ T, such that $\vdash_{\overline{\kappa}_0} (\phi_1 \wedge \cdots \wedge \phi_n) \rightarrow \phi$. Note that due to the definedness convention if T $\vdash_{\overline{\kappa}_0} \phi$ then Vocab(T) $\subseteq$ Vocab.

Consequences

Some simple theorems and derivable rules of the system are:

(C) $\quad \vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \wedge \mathtt{ist}(\kappa_1, \psi) \rightarrow \mathtt{ist}(\kappa_1, \phi \wedge \psi)$

(Or) $\quad \vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \vee \mathtt{ist}(\kappa_1, \psi) \rightarrow \mathtt{ist}(\kappa_1, \phi \vee \psi)$

(M) $\quad \vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi \wedge \psi) \rightarrow \mathtt{ist}(\kappa_1, \phi) \wedge \mathtt{ist}(\kappa_1, \psi)$

(K*) $\quad \vdash_{\overline{\kappa}} \mathtt{ist}(\overline{c}, \phi \rightarrow \psi) \rightarrow \mathtt{ist}(\overline{c}, \phi) \rightarrow \mathtt{ist}(\overline{c}, \psi)$

(REP) $$\frac{\vdash_{\overline{\kappa}} \phi_1 \leftrightarrow \phi'_1 \cdots \vdash_{\overline{\kappa}} \phi_n \leftrightarrow \phi'_n}{\vdash_{\overline{\kappa}} \psi(\phi_1, \ldots, \phi_n) \leftrightarrow \psi(\phi'_1, \ldots, \phi'_n)}$$

provided $\psi(p_1, \ldots p_n) \in$ PROP.

A slightly deeper result is that any formula is provably equivalent to one in a certain syntactic form. This equivalence plays an important role in the completeness proof.

Definition (CNF): A formula ϕ is in conjunctive normal form (CNF) iff it is of the form $E_1 \wedge E_2 \wedge \cdots \wedge E_k$, and each E_i is of the form $\alpha_{i1} \vee \alpha_{i2} \vee \cdots \vee \alpha_{ir_i}$, where each α_{ij} is either a literal, or $\mathtt{ist}^{\pm}(\overline{c}, \beta)$ for some disjunction of literals β. Note that i and k can be 1.

Lemma (CNF): For any formula ϕ, context sequence $\overline{\kappa}$, there exists a formula ϕ^* which is in CNF, such that $\vdash_{\overline{\kappa}} \phi \leftrightarrow \phi^*$.

(PL)	$\vdash_{\overline{\kappa}} \phi$	provided ϕ is an instance of a tautology.
(K)	$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi \to \psi) \to \mathtt{ist}(\kappa_1, \phi) \to \mathtt{ist}(\kappa_1, \psi)$	
$(\triangle_+)$	$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \vee \psi) \to \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \vee \mathtt{ist}(\kappa_1, \psi)$	
$(\triangle_-)$	$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa_2, \phi) \vee \psi) \to \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa_2, \phi)) \vee \mathtt{ist}(\kappa_1, \psi)$	
(MP)	$\dfrac{\vdash_{\overline{\kappa}} \phi \quad \vdash_{\overline{\kappa}} \phi \to \psi}{\vdash_{\overline{\kappa}} \psi}$	(CS) $\dfrac{\vdash_{\overline{\kappa}*\kappa_1} \phi}{\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi)}$

Table 1: Axioms and Inference Rules

Theorem (soundness): If $\vdash_{\overline{\kappa}} \phi$, then for all models $\mathfrak{M}$ classical on Vocab $\mathfrak{M} \models_{\overline{\kappa}} \phi$. If T $\vdash_{\overline{\kappa}} \phi$, then for all models $\mathfrak{M}$ classical Vocab if for all $\psi \in$ T $\mathfrak{M} \models_{\overline{\kappa}} \psi$, then $\mathfrak{M} \models_{\overline{\kappa}} \phi$.

Completeness

We begin by introducing some concepts needed to state the completeness theorem.

Definition (satisfiability): A set of formulas T is *satisfiable in context* $\overline{\kappa}$ *with vocabulary* Vocab iff there exists a model $\mathfrak{M}$ classical on Vocab, such that for all $\phi \in$ T, $\mathfrak{M} \models_{\overline{\kappa}} \phi$.

Definition (consistency): A formula ϕ is *consistent in* $\overline{\kappa}$ *with* Vocab, where $\mathrm{Vocab}(\overline{\kappa}, \phi) \subseteq$ Vocab iff not $\vdash_{\overline{\kappa}} \neg\phi$. A finite set T is consistent in $\overline{\kappa}$ with Vocab iff $\bigwedge$ T is consistent in $\overline{\kappa}$ with Vocab. An infinite set T is consistent in $\overline{\kappa}$ with Vocab iff every finite subset of T is consistent in $\overline{\kappa}$ with Vocab. A set T is inconsistent in $\overline{\kappa}$ with Vocab iff the set T is not consistent in $\overline{\kappa}$ with Vocab.

A set T is maximally consistent in $\overline{\kappa}$ with Vocab iff T is consistent in $\overline{\kappa}$ with Vocab and for all $\phi \notin$ T such that $\mathrm{Vocab}(\overline{\kappa}, \phi) \subseteq$ Vocab, T $\cup \{\phi\}$ is inconsistent in $\overline{\kappa}$ with Vocab.

As is usual, an important part of the completeness proof is the Lindenbaum lemma allowing any consistent set of wffs to be extended to a maximally consistent set.

Lemma (Lindenbaum): If T is consistent in $\overline{\kappa}$ with Vocab, then T can be extended to a maximally consistent set T_0 in $\overline{\kappa}$ with Vocab.

Now we proceed to state and prove the completeness of the system.

Theorem (completeness): For any set of formulas T, T is consistent in $\overline{\kappa}_0$ with Vocab iff T is satisfiable in $\overline{\kappa}_0$ with Vocab.

Proof (completeness): Assume T is consistent in $\overline{\kappa}_0$ with Vocab. By the (**Lindenbaum lemma**) we can extend T to a maximally consistent set T_0. From T_0 we will construct the model $\mathfrak{M}_0$. For each $\overline{\kappa} = \overline{\kappa}_0 * \overline{c} \in \mathbb{K}^*$ define

$$\mathrm{T}_{\overline{\kappa}+} := \{\phi \mid \mathrm{T}_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \phi), \phi \in \mathrm{PROP}\}.$$

Lemma ($\mathrm{T}_{\overline{\kappa}+}$): $\mathrm{T}_{\overline{\kappa}+}$ is closed under logical consequence: for all ϕ where $\mathrm{Vocab}(\overline{\kappa}, \phi) \subseteq$ Vocab, if ϕ tautologically follows from $\mathrm{T}_{\overline{\kappa}+}$ then $\phi \in \mathrm{T}_{\overline{\kappa}+}$.

Note that $\mathrm{T}_{\overline{\kappa}+}$ need not be either maximally consistent or even consistent. Now, using only the sets $\mathrm{T}_{\overline{\kappa}+}$ of formulas from PROP, we will define a model $\mathfrak{M}_0$ for the set of formulas T_0. We define the domain of $\mathfrak{M}_0$

$$\mathrm{Dom}(\mathfrak{M}_0) := \{\overline{\kappa} \mid \overline{\kappa} \leq \overline{\kappa}_0, \exists \overline{\kappa}' \in \mathrm{Dom}(\mathsf{Vocab}), \overline{\kappa}' \leq \overline{\kappa}\}$$

and for all $\overline{\kappa} \in \mathrm{Dom}(\mathfrak{M}_0)$

$$\mathfrak{M}_0(\overline{\kappa}) := \{\nu \mid \mathrm{Dom}(\nu) = \mathsf{Vocab}(\overline{\kappa}), \forall \phi \in \mathrm{T}_{\overline{\kappa}+}, \overline{\nu}(\phi) = 1\}.$$

In the above, $\overline{\nu}$ is the unique homomorphic extension of ν with respect to the propositional connectives. To see that $\mathfrak{M}_0$ as defined is a model, we first note that it clearly meets condition 1, since all the truth assignments associated with a context must have the same domain. Condition 2 is met since $\mathrm{Dom}(\mathfrak{M}_0)$ as defined is a subtree rooted at $\overline{\kappa}_0$. Note that if $\mathrm{T}_{\overline{\kappa}+}$ is empty (which corresponds to the case where $\mathsf{Vocab}(\overline{\kappa}) = \emptyset$), then $\mathfrak{M}_0(\overline{\kappa})$ is a singleton set, whose only member is the *empty truth assignment*. Finally, to establish completeness we need only prove the truth lemma. The proof of the truth lemma is based on the CNF construction and is the novel aspect of this completeness proof.

Lemma (truth):
For any ϕ such that $\mathrm{Vocab}(\overline{\kappa}_0, \phi) \subseteq$ Vocab,

$$\phi \in \mathrm{T}_0 \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi.$$

Clearly, if $\phi \in$ T then also $\phi \in \mathrm{T}_0$ and therefore by truth lemma we get $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi$. $\Box_{\textbf{completeness}}$

Before we give the proof of the truth lemma, we need to state a property of the model $\mathfrak{M}_0$ which is needed in the ist case of the truth lemma.

Lemma ($\mathfrak{M}_0$): Let $\mathfrak{M}_0$ be a model as defined from T_0 in the completeness proof. Then for all $\phi \in$ PROP where $\mathrm{Vocab}(\overline{\kappa}_0 * \overline{c}, \phi) \subseteq \mathsf{Vocab}$,

$$T_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \phi) \text{ iff for all } \nu \in \mathfrak{M}_0(\overline{\kappa}_0 * \overline{c}) \quad \nu(\phi) = 1.$$

A frequently used instance of the $\mathfrak{M}_0$ lemma is that $T_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \phi \wedge \neg\phi)$ iff $\mathfrak{M}_0(\overline{\kappa}_0 * \overline{c}) = \emptyset$, for all ϕ satisfying the (**definedness condition**).

Proof (truth lemma): Instead of proving $\phi \in T_0$ iff $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi$ we will prove the statement

(TL) $\quad \psi$ is in CNF implies $(\psi \in T_0$ iff $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \psi)$.

To see that the former follows from the latter, assume $\phi \in T_0$. By the (**CNF lemma**), there exists formula ϕ^* in CNF such that $\vdash_{\overline{\kappa}_0} \phi \leftrightarrow \phi^*$. Using maximal consistency of T_0, it follows that $\phi^* \in T_0$. Therefore by (**TL**) it must be the case that $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi^*$. Our logic is sound: $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi^*$ iff $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi$, and thus we conclude that $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi$. We can simply reverse the steps of the argument to prove the other direction of the biconditional.

We prove the (**TL**) by induction on the structure of the formula ψ. In the base case ψ is an atom, and thus in CNF. From the definition of $\mathfrak{M}_0(\overline{\kappa}_0)$ it follows that $\rho \in T_0 \Leftrightarrow \mathfrak{M}_0 \models_{\overline{\kappa}_0} \rho$. In proving the inductive step we first examine $\psi = \chi \vee \mu$. The inductive hypothesis is that the lemma is true for formulas χ and μ. Assume $\chi \vee \mu$ is in CNF. Then both χ and μ must also be in CNF. Since T_0 is maximally consistent $\chi \vee \mu \in T_0$ iff either $\chi \in T_0$ or $\mu \in T_0$. By the inductive hypothesis this will be true iff either $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \chi$ or $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mu$, and by the definition of satisfaction iff $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \chi \vee \mu$. The inductive step for conjunction and negation is similar. We make use of the fact that if $\chi \wedge \mu$ is in CNF, then so are both χ and μ; and if $\neg\chi$ is in CNF, then so is χ. The interesting case is when ψ is an ist. Assume that ψ is in CNF. Then ψ must be of the form

$$\psi = \mathtt{ist}^{\pm}(\overline{c}, \chi),$$

where χ is a disjunction of literals. The context sequence $\overline{c}$ will sometimes be written as $\kappa_1 * \cdots * \kappa_n$. We will examine two cases, depending on whether or not any of the sets of sentences $T_{(\overline{\kappa}_0 * \overline{c}')+}$ where $\overline{c} \leq \overline{c}'$, are inconsistent. The sets $T_{(\overline{\kappa}_0 * \overline{c}')+}$, where $\overline{c} \leq \overline{c}'$, are all consistent iff the formula

$$(\mathrm{D}_{\overline{c}}) \quad \mathtt{ist}(\overline{c}, \neg\phi) \rightarrow \neg\mathtt{ist}(\overline{c}, \phi)$$

is in T_0, for any wff ϕ which satisfies the definedness condition. The proof of this is identical to the soundness and completeness proofs of a context system with axiom schema (**D**) w.r.t. the set of consistent models, dealt with shortly. Formula $(\mathrm{D}_{\overline{c}})$ is equivalent to

$$\neg\mathtt{ist}(\overline{c}, \phi \wedge \neg\phi) \in T_0,$$

for all ϕ satisfying the definedness condition; the proof carries over from normal systems of modal logic. Now we state a useful consequence of $(\mathrm{D}_{\overline{c}})$'s.

Lemma ($\mathrm{D}_{\overline{c}}$):
Let $\overline{c}$ be $\kappa_1 * \cdots * \kappa_n$. If $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{n-1})} \in T_0$, then

$$\mathtt{ist}^{\pm}(\overline{c}, \phi) \in T_0 \quad \text{iff} \quad \pm\,\mathtt{ist}(\overline{c}, \phi) \in T_0$$

for any formula ϕ which satisfies the definedness convention. The sign on the right hand side is positive iff there is an even number of negations in the $\mathtt{ist}^{\pm}$ on the left hand side.

Now we examine the two cases need to prove the inductive step for ist of the truth lemma.

Case $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{n-1})} \in T_0$: In this case we assume $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{n-1})} \in T_0$ and that $\psi \in T_0$. Then by the $\mathrm{D}_{\overline{c}}$ lemma:

$$\mathtt{ist}^{\pm}(\overline{c}, \chi) \in T_0 \quad \text{iff} \quad \pm\,\mathtt{ist}(\overline{c}, \chi) \in T_0$$

We only include the positive case.

$$\mathtt{ist}(\overline{c}, \chi) \in T_0 \quad \text{iff} \quad T_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi)$$

Now by ($\mathfrak{M}_0$ **lemma**) and the definedness condition $\mathrm{Vocab}(\overline{\kappa}_0 * \overline{c}) \subseteq \mathsf{Vocab}$ we have

$$T_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi) \quad \text{iff} \quad (\forall \nu \in \mathfrak{M}_0(\overline{\kappa}))(\overline{\nu}(\chi) = 1)$$

By the definition of satisfaction:

$$(\forall \nu \in \mathfrak{M}_0(\overline{\kappa}))(\overline{\nu}(\chi) = 1) \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi)$$

Now since $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{n-1})} \in T_0$, and by ($\mathfrak{M}_0$ **lemma**) we obtain:

$$\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi) \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}^{\pm}(\overline{c}, \chi)$$

Case $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{n-1})} \notin T_0$: Let j be the index of the first inconsistent context; formally $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_j)} \notin T_0$ and $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{j-1})} \in T_0$. Then for all ϕ satisfying the definedness condition we have $\neg\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi \wedge \neg\phi) \notin T_0$. Now by maximal consistency of T_0, (K^*) and (**MP**)

$$\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi \wedge \neg\phi) \in T_0 \text{ iff } \mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \psi) \in T_0$$

Thus, $T_{(\overline{\kappa}_0 * \kappa_1 * \cdots * \kappa_j)+}$ is inconsistent, $\mathfrak{M}_0(\overline{\kappa}_0 * \kappa_1 * \cdots * \kappa_j) = \emptyset$, and consequently

$$\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi) \in T_0 \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi)$$

for all ϕ such that $\mathrm{Vocab}(\overline{\kappa}_0 * \kappa_1 * \cdots * \kappa_j, \phi) \subseteq \mathsf{Vocab}$. Then by reasoning similar to the previous case we get:

$$\mathtt{ist}^{\pm}(\overline{c}, \chi) \in T_0 \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}^{\pm}(\overline{c}, \chi).$$

Note that in the entire proof of the inductive step for ist, we did not need the inductive hypothesis, making use only of the special form of χ which is guaranteed because ψ is in CNF. $\Box_{\textbf{truth-lemma}}$

Correspondence Results

In this section we provide soundness and completeness results for several extensions of the general system. correspond to certain intuitive principles concerning the nature of contexts. In each extension the syntax and semantics is the same as in the general case, and the (**definedness convention**) still holds. Only the class of models and axioms are modified.

Consistency

Sometimes it is desirable to ensure that all contexts are consistent.

In this system we examine the class, $\mathfrak{Consistent}$, of *consistent models.* A model $\mathfrak{M} \in \mathfrak{Consistent}$ iff for any context sequence $\overline{\kappa} \in \mathrm{Dom}(\mathfrak{M})$,

$$\mathfrak{M}(\overline{\kappa}) \neq \emptyset.$$

The following axiom schema is sound with respect to the class of consistent models $\mathfrak{Consistent}$:

(D) $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa, \neg\phi) \rightarrow \neg\mathtt{ist}(\kappa, \phi)$

Axiom schema (D) is also commonly used in modal logic, and is sound and complete for the set of serial Kripke frames, in which for each world there is another world from which it is accessible from. Note that axiom (D) is equivalent to

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa, \phi \wedge \neg\phi).$$

Theorem (completeness): The general context system with (D) axiom schema is complete with respect to the set of models $\mathfrak{Consistent}$.

Flatness

For some applications all contexts will be identical regardless of where they are examined from. This type of situation will often arise when we use a number of independent databases. For example, if I am booked on flight 921 in the context of the Northwest airlines database, then regardless of which travel agent I choose, in the context of that travel agent, it is true that in the context of Northwest airlines I am booked on flight 921.

In this system we examine a class, $\mathfrak{Flat}$, of what we call *flat models.* A model $\mathfrak{M}$ is flat, formally $\mathfrak{M} \in \mathfrak{Flat}$ iff $\mathrm{Dom}(\mathfrak{M}) = \mathbb{K}^*$ and for any context sequences $\overline{\kappa}_1$ and $\overline{\kappa}_2$, and any context κ,

$$\mathfrak{M}(\overline{\kappa}_1 * \kappa) = \mathfrak{M}(\overline{\kappa}_2 * \kappa).$$

When dealing with flat models it might be more intuitive to think of individual contexts rather then context sequences. Then $\mathfrak{M} \in \mathfrak{Flat}$ can be viewed as a function which maps contexts to finite sets of partial truth assignments, in other words

$$\mathfrak{M} \in \mathbb{K} \cup \{\epsilon\} \mapsto \mathbf{P}(\mathbb{P} \rightarrow_{\mathrm{p}} 2).$$

with the side condition of general models that still applies:

$$(\forall \overline{\kappa} \in \mathbb{K} \cup \{\epsilon\})(\forall \nu_1, \nu_2 \in \mathfrak{M}(\overline{\kappa}))(\mathrm{Dom}(\nu_1) = \mathrm{Dom}(\nu_2))$$

The following flatness axiom schemas are sound with respect to the class of flat models $\mathfrak{Flat}$:

(Fl_+) $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_2, \mathtt{ist}(\kappa_1, \phi)) \leftrightarrow \mathtt{ist}(\kappa_1, \phi)$

(Fl_-) $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_2, \neg\mathtt{ist}(\kappa_1, \phi)) \rightarrow \neg\mathtt{ist}(\kappa_1, \phi)$

providing the vocabulary also satisfies the flatness condition: for any context sequences $\overline{\kappa}_1$ and $\overline{\kappa}_2$, and any context κ,

$$\mathsf{Vocab}(\overline{\kappa}_1 * \kappa) = \mathsf{Vocab}(\overline{\kappa}_2 * \kappa).$$

The backward direction of the flatness axiom schemas ($\mathbf{Fl}_+$) corresponds of the modal logic axiom schema S4 (provided that κ_1 is the same as κ_2). Similarly, the converse of ($\mathbf{Fl}_-$) corresponds to the modal logic axiom schema S5. Note that the converse of ($\mathbf{Fl}_-$) is a theorem in the system.

It is interesting to observe that in every system with ($\mathbf{Fl}_+$) and ($\mathbf{Fl}_-$), (**D**) is also derivable. In semantic terms, this means that any flat model is also a consistent model; a reasonable property for if a context was inconsistent, then in that context it would be true that all other contexts are also inconsistent. Due to flatness, this would really make all the other contexts inconsistent.

Theorem (completeness): The general context system with (Fl_+) and (Fl_-) axiom schemas is complete with respect to the set of flat models $\mathfrak{Flat}$.

Truth

It might be more intuitive to define the `ist` modality to correspond to truth rather than validity; incidently this is also where the `ist` predicate got its name: `is true`. Truth based interpretation of the basic context modality also corresponds to the original suggestions by McCarthy [McCarthy, 1993]. In this case a context is associated with a single truth assignment rather than a set of truth assignments.

We examine the class, $\mathfrak{Truth}$, of *truth models.* A model $\mathfrak{M}$ is a truth model, formally $\mathfrak{M} \in \mathfrak{Truth}$ iff for any context sequence $\overline{\kappa} \in \mathrm{Dom}(\mathfrak{M})$,

$$|\,\mathfrak{M}(\overline{\kappa})\,| \leq 1.$$

The following axiom schema is sound with respect to the class of truth models $\mathfrak{Truth}$:

(Tr) $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa, \phi) \vee \mathtt{ist}(\kappa, \neg\phi)$

Note that (**Tr**) is the converse of (**D**).

Theorem (completeness): The general context system with (**Tr**) axiom schema is complete with respect to the set of truth models $\mathfrak{Truth}$.

Previously we said that $(\triangle_+)$ and $(\triangle_-)$ are derivable in a system which contains (**D**) and (**Tr**). In fact, a stronger formula is true of this system:

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa, \phi \vee \psi) \leftrightarrow (\mathtt{ist}(\kappa, \phi) \vee \mathtt{ist}(\kappa, \psi)).$$

Meaninglessness as Falsity

In this section we examine a slightly more elaborate modification of the general system. This modification closely models the semantics described, but not investigated, in [Guha, 1991]. The general idea here is that if ϕ is not in the vocabulary of κ, then $\mathtt{ist}(\kappa, \phi)$ is taken to be false instead of meaningless or undefined. To cater faithfully to this interpretation, two changes must be made to the semantics of the general system. Firstly, the ist clause in the definition of Vocab : $\mathbb{K}^* \times \mathbb{W} \rightarrow \mathbf{P}(\mathbb{K}^* \times \mathbb{P})$ must be altered to reflect the fact that $\mathtt{ist}(\kappa, \phi)$ will always be in the vocabulary of any context. Secondly, the ist clause in the definition of satisfaction must also be modified. The appropriate new clause in the definition of Vocab is:

$$\text{Vocab}(\overline{\kappa}, \phi) = \emptyset \text{ if } \phi \text{ is } \mathtt{ist}(\kappa, \phi_0)$$

While the new clause in the definition of satisfaction is:

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \text{ iff } \text{Vocab}(\phi, \overline{\kappa} * \kappa_1) \subseteq \mathit{Vocab}(\mathfrak{M}) \text{ and for all } \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}} \quad \mathfrak{M}, \nu_1 \models_{\overline{\kappa} * \kappa_1} \phi$$

The other clauses in both definitions remain the same, modulo the fact that all occurrences of Vocab in the definition of satisfaction now refer to the new definition. We maintain the (**definedness convention**) in stating the proof system for this version, but again we point out that all occurrences of Vocab now refers to the new definition. The proof system for this version consists of the axioms and rules of the general system, together with the new axiom:

$$\text{(MF)} \quad \vdash_{\overline{\kappa}} \neg\mathtt{ist}(\kappa_1, \phi) \qquad \text{if Vocab}(\overline{\kappa} * \kappa_1, \phi) \not\subseteq \mathsf{Vocab}$$

The completeness proof for this system is structurally similar to the one described in this paper. The only new points are those that arise out of the liberal definition of Vocab.

Related Work

Our work is largely based on McCarthy's ideas on context. McCarthy's research [McCarthy, 1987; McCarthy, 1993] in formalizing common sense has led him to believe that in order to achieve human-like generality in reasoning, we need to develop a formal theory of context. The key idea in McCarthy's proposal was to to treat contexts as formal objects, which enables one to state that a proposition is true in a context: $\mathtt{ist}(\kappa, \phi)$ where ϕ is a proposition and κ is a context. This permits axiomatizations in a limited context to be expanded so as to *transcend* their original limitations.

There has been other research done in this area, most notable is the work of Lifschitz, Shoham, and Guha. We briefly treat each in turn.

Two contexts can differ in, at least, three ways: they may have different vocabularies; or they may have the same vocabulary but describe different states of affairs, or (in the first order case) they may have the same vocabulary (i.e. language) but treat it differently (i.e the arities may not be the same). The first two differences were studied in [Buvač, 1992], and led to two different views on the use of context. Lifschitz's early note on formalizing context [Lifschitz, 1986] concentrates on the third difference. Shoham, in his work on contexts, concentrates on the second difference [Shoham, 1991]. Every proposition is meaningful in every context, but the same proposition can have different truth values in different contexts. Shoham approached the task of formalizing context from the perspective of modal and non-classical of logics. He defines a propositional language with an analogue to the ist modality, and a relation $\kappa_1 \bullet\!\supset \kappa_2$, expressing that context κ_1 is as general as context κ_2. Drawing on the intuitive analogy between a context κ and the proposition *current-context*(κ), Shoham identifies the set of contexts with the set of propositions. This enables him to define truth in a context $\mathtt{ist}(\kappa, p)$, in terms of the the conditional *current-context*(κ) $\rightarrow p$, where $\rightarrow$ is interpreted as as some form of intuitionistic or relevance implication. His paper gives a list of 14 benchmark sentences which characterize this implication.

Guha's dissertation contains a number of examples of context use. These demonstrate how reasoning with contexts should behave, and which properties a formalization of context should exhibit. The Cyc knowledge base [Guha and Lenat, 1990], which is the main motivation for Guha's context research, is made up of many theories, called *micro-theories*, describing different aspects of the world. Guha has tailored the design of micro-theories after contexts.

There is also a clear parallel between the logic of context and the modal logics of knowledge and belief [Halpern and Moses, 1992]. The modality $\mathtt{ist}(\kappa, \phi)$ may be interpreted as expressing that the agent κ knows or believes the sentence ϕ. In the case where there is only one context, our formal system collapses to a normal system of modal logic (with two additional axiom schemas $(\triangle_+)$ and $(\triangle_-)$). This is analogous to the way logics of knowledge and belief collapse to a normal system of modal logic in case of a single agent. However, the logics of knowledge and belief differ from our logic of contexts in a number of ways: Firstly, logics of knowledge and belief do not deal with variable vocabularies and the corresponding partiality. Furthermore, logics of knowledge and belief are usually ascribed possible world semantics. Consequently, an agent's belief is modeled by relations between worlds. Modeling truth or validity in a context by a relation between worlds would not be intuitive because we want contexts to be reified as first class objects in the semantics. This will allow us (in the predicate case) to state relations between contexts, define operations on contexts, and specify how sentences from one context can be *lifted* into another context.

Conclusions and Future Work

Our goal is to extend the system to a full quantification logic. One advantage of quantificational system is that it enables us to express relations between context, operations on contexts, and state *lifting rules* which describe how a fact from one context can be used in another context. However, in the presence of context variables it might not be possible to define the vocabulary of a sentence without knowing which object a variable is bound to. Therefore the first step in this direction is to to examine propositional systems with dynamic definitions of meaningfulness.

We also plan to define non-Hilbert style formal systems for context. Probably the most relevant is a natural deduction system, which would be in line with McCarthy's original proposal of treating contextual reasoning as a strong version of natural deduction. In such a system, entering a context would correspond to making an assumption in natural deduction, while exiting a context corresponds to discharging an assumption.

Finally, it would be interesting to show some formal properties of our logic. These include defining a decision procedure, in the style of [Mints, 1992].

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