The Semantics of Propositional Contexts

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Abstract. In this paper we investigate the semantic properties of *contexts*. We describe the syntax and semantics of the propositional logic of context. This logic extends classical propositional logic in two ways. Firstly, a new modality, $\mathtt{ist}(\kappa, \phi)$, is introduced. It is used to express that the sentence, ϕ, holds in the context κ. Secondly, each context has its own vocabulary, i.e. a set of propositional atoms which are *defined* or *meaningful* in that context. The main results of this paper are a proof that our logic is decidable and comparison of our semantics to Kripke semantics.

1 Introduction

In this paper we investigate the semantic properties of *contexts* as they appear in declarative AI. Contexts were first suggested in McCarthy's Turing Award Paper, [5], as a possible solution to the problem of generality in AI. Our main motivation for formalizing contexts is to solve this problem. We want to be able to make AI systems which are never permanently stuck with the concepts they use at a given time because they can always transcend the context they are in. Such a capability would allow the designer of a reasoning system to include only such phenomena as are required for the system's immediate purpose, retaining the assurance that if a broader system is required later, "lifting axioms" can be devised to restate the facts from the narrow context in the broader context with qualifications added as necessary. We provide two simple examples.

The first example is due to McCarthy [7]. It illustrates how a reasoning system can utilize contexts to incorporate information from a general common sense knowledge base into other specialized knowledge bases. Assume that in the context of situation calculus $on(x, y, s)$ is used to express the fact that object x is on top of object y in situation s. Although no mention to the notion of *above* is made in the context of situation calculus, we are interested to know which of the *above* relations hold in a particular situation. The definition of *above* in terms of *on* is likely to be found in some context of general common sense knowledge. The context formalism will allow a reasoning system to use the theory of situation calculus and the theory of general common sense knowledge together. Furthermore, in the logic we can write axioms to import or *lift* the definition of *above* from the context of general common sense knowledge into the context of situation calculus. Although *above* was not originally defined in the context of situation calculus, the system, after lifting, will be able to infer which *above* relations hold in a particular situation. Of course, the power of a full quantificational logic will be needed to adequately address this example.

The second example concerns theories which were not originally intended to be used together, and in fact might, on the surface, seem inconsistent. For example, assume a common sense knowledge base of Stanford University contains the proposition "kids drive BMW's". A common sense knowledge base of Berkeley, which was not originally intended to be used with the above mentioned Stanford knowledge base, will probably contain the negation of this proposition. A logic of context will enable a reasoning system to use such seemingly inconsistent knowledge bases without deriving a contradiction.

Although the notion of context has existed in philosophy, linguistics, and natural language processing for some time, our work is based on the idea of treating context as a *formal object.* This means that a context is an object in the semantics denoted by a corresponding constant in the language. The language also contains the `ist` modality (pronounced as "is true"), which allows us to talk about a sentence being true in a context. For example, $\mathtt{ist}(\kappa, p)$, means that the sentence p is true in the context κ. In the semantics, contexts are associated with a set of truth assignments, reflecting the states of affairs in that context. Proposition p is true in context κ if every truth assignment associated with κ satisfies p.

A formal logical explication of contexts was first given in [4]. In that paper we describe both the syntax and semantics of a general propositional language of context, and give a Hilbert style proof system for this language. The main results of that paper were the soundness and completeness of the proof system. We also provided soundness and completeness results (i.e. correspondence theory) for various extensions of the basic system.

The aim of this paper is to answer some semantic questions not resolved by the above mentioned paper. These are: (1) What is the relation between the semantics of context and Kripke semantics? (2) Is the system decidable, i.e. does the logic have the finite model property? Due to lack of space, proofs of the theorems are not included in this paper. These, as well as additional discussions on the subject and related works can be found in [3].

1.1 Notation

We use standard mathematical notation. If X and Y are sets, then $X \to_{\mathrm{p}} Y$ is the set of partial functions from X to Y. $\mathbf{P}(X)$ is the set of subsets of X. X^* is the set of all finite sequences, and we let $\bar{x} = [x_1, \ldots, x_n]$ range over X^*. ϵ is the empty sequence. We use the infix operator $*$ for appending sequences. We make no distinction between an element and the singleton sequence containing that element. Thus we write $\bar{x} * x_1$ instead of $\bar{x} * [x_1]$. As is usual in logic we treat X^* as a tree (that grows downward). $\bar{x}_1 < \bar{x}_0 \leq \epsilon$ iff $\bar{x}_1$ properly extends $\bar{x}_0$ (i.e.

$(\exists \bar{y} \in X^* - \{\epsilon\})(\bar{x}_1 = \bar{x}_0 * \bar{y}))$. We say $Y \subseteq X^*$ is a subtree rooted at $\bar{y}$ to mean (1) $\bar{y} \in Y$ and $(\forall \bar{z} \in Y)(\bar{z} \leq \bar{y})$ and (2) $(\forall \bar{z} \in Y)(\forall \bar{w} \in X^*)(\bar{z} \leq \bar{w} \leq \bar{y} \rightarrow \bar{w} \in Y)$.

2 The Propositional Logic of Context

The propositional logic of context extends classical propositional logic in two ways. Firstly, a new modality, $\mathtt{ist}(\kappa, \phi)$, is introduced. It is used to express that the sentence, ϕ, holds in the context κ. Secondly, each context has its own vocabulary, i.e. a set of propositional atoms which are *defined* or *meaningful* in that context. The vocabulary of one context may or may not overlap with another context.

2.1 Syntax

We begin with two distinct countably infinite sets, $\mathbb{K}$ the set of all contexts, and $\mathbb{P}$ the set of propositional atoms. The set, $\mathbb{W}$, of well-formed formulas (wffs) is built up from the propositional atoms, $\mathbb{P}$, using the usual propositional connectives (negation and implication) together with the `ist` modality.

Definition ($\mathbb{W}$): $\mathbb{W} = \mathbb{P} \cup (\neg\mathbb{W}) \cup (\mathbb{W} \rightarrow \mathbb{W}) \cup \mathtt{ist}(\mathbb{K}, \mathbb{W})$
The operations $\wedge$, $\vee$ and $\leftrightarrow$ are defined as abbreviations in the usual way.

We have chosen to develop a modal logic rather than to reifying sentences and treat `ist` as a regular predicate, because (1) it leads to a more natural semantics (defined in the following subsection), and (2) the language does not allow self-referential statements, thus avoiding paradoxes. Although self-referential formulas are relevant for developing theories of truth, they are not needed for describing states of affairs which hold in particular contexts. Therefore, the loss of expressive power due to lack of self-referential formulas will not be missed in our logic. The latter approach, of reifying formulas, is taken by Attardi and Simi in [1]. We further discuss their work in and its relation to our system in [3].

2.2 Semantics

To explain the semantics we first introduces a naïve notion of a model, which is then refined in two stages.

Naïvely, a context is modelled by a set of truth assignments, that describe the possible states of affairs of that context. Thus a model will associate a set of truth assignments with every context. These truth assignments reflect the states of affairs which are possible in a context. For a proposition to be true in a context it has to be satisfied by all the truth assignments associated with that context. Therefore, the `ist` modality is interpreted as validity: $\mathtt{ist}(\kappa, \rho)$ is true iff the propositional atom ρ is true in all the truth assignments associated with context κ. Treatment of `ist` as validity corresponds to Guha's proposal for context semantics, which was motivated by the Cyc knowledge base. A system which models a context by a single truth assignment, thus interpreting `ist` as truth, can be obtained by placing simple restrictions on the definition of a model and by enriching the set of axioms.

However, this naïve model is not powerful enough to represent some properties desired of contexts. Therefore, we need to refine our naïve notion of a model. We do this in two stages.

Firstly, the nature of a particular context may itself be context dependent. For example, in the context of the 1950's, the context of car racing is different from the context of car racing viewed from today's context. This naturally leads to considering sequences of contexts rather than a context in isolation. So, a model will associate a set of truth assignments with a context sequence, rather than an individual context (as was the case in the naïve view). We refer to this feature of the system as *non-flatness*. It reflects on the intuition that what holds in a context can depend on how this context has been reached, i.e. from which perspective it is being viewed. For example, non-flatness will be desirable if we represent the beliefs of an agent as the sentences which hold in a context. A system of flat contexts can easily be obtained by placing certain restrictions on what kinds of structures are allowed as models, as well as enriching the axiom system (cf. §3.2 in [3]).

Secondly, since different contexts can have different vocabularies, some propositions can be meaningless in some contexts, and therefore the truth assignments describing the state of affairs in that context need to be partial.

Now we are ready to define the general model:

Definition ($\mathfrak{M}$): In this system a model, $\mathfrak{M}$, will be a function which maps a context sequence $\overline{\kappa} \in \mathbb{K}^*$ to a set of partial truth assignments,

$$\mathfrak{M} \in (\mathbb{K}^* \rightarrow_{\mathrm{p}} \mathbf{P}(\mathbb{P} \rightarrow_{\mathrm{p}} 2)),$$

with the added conditions that

1. $(\forall \overline{\kappa})(\forall \nu_1, \nu_2 \in \mathfrak{M}(\overline{\kappa}))(\mathrm{Dom}(\nu_1) = \mathrm{Dom}(\nu_2))$
2. $\mathrm{Dom}(\mathfrak{M})$ is a subtree of $\mathbb{K}^*$ rooted at some context sequence $\overline{\kappa}_0$.

We write $\overline{\kappa}^{\mathfrak{M}}$ to denote the set of partial truth assignments $\mathfrak{M}(\overline{\kappa})$. Note that $\overline{\kappa}^{\mathfrak{M}}$ can be empty. Since all the elements of $\mathfrak{M}(\overline{\kappa})$ have the same domain, which is imposed by condition 1. above, we will write $\mathrm{Dom}(\mathfrak{M}(\overline{\kappa}))$ to refer to this domain. The collection of all such models will be denoted by $\mathbb{M}$.

Vocabularies To capture the intuition that different contexts can have different vocabularies, we make the truth assignments in our model partial. The atoms which are given a truth value in a context sequence are defined by a relation $\mathsf{Vocab} \subseteq \mathbb{K}^* \times \mathbb{P}$. Given a Vocab, the *vocabulary of a context sequence* $\overline{\kappa}$, or the set of atoms which are meaningful in that sequence, is $\{\rho \mid <\overline{\kappa}, \rho> \in \mathsf{Vocab}\}$.

Definition (*Vocab* of $\mathfrak{M}$): We define a function $\mathit{Vocab} : \mathbb{M} \rightarrow \mathbf{P}(\mathbb{K}^* \times \mathbb{P})$, which given a model returns the vocabulary of the model:

$$\mathit{Vocab}(\mathfrak{M}) := \{<\overline{\kappa}, \rho> \mid \overline{\kappa} \in \mathrm{Dom}(\mathfrak{M}) \text{ and } \rho \in \mathrm{Dom}(\mathfrak{M}(\overline{\kappa}))\}$$

We say that a model $\mathfrak{M}$ is *classical on vocabulary* Vocab iff Vocab $\subseteq$ *Vocab*($\mathfrak{M}$).

The notion of vocabulary can also be applied to sentences. Intuitively, the vocabulary of a sentence relates a context sequence to the atoms which occur in the scope of that context sequence. In the definition we also need to take into account that sentences are not given in isolation but in a context.

Definition (Vocab **of** ϕ **in** $\overline{\kappa}$)**:** We define a function Vocab : $\mathbb{K}^* \times \mathbb{W} \rightarrow \mathbf{P}(\mathbb{K}^* \times \mathbb{P})$ which given formula in a context, returns the vocabulary of the formula.

$$\text{Vocab}(\overline{\kappa}, \phi) = \begin{cases} \{<\overline{\kappa}, \phi>\} & \phi \in \mathbb{P} \\ \text{Vocab}(\overline{\kappa}, \phi_0) & \phi \text{ is } \neg\phi_0 \\ \text{Vocab}(\overline{\kappa} * \kappa, \phi_0) & \phi \text{ is } \texttt{ist}(\kappa, \phi_0) \\ \text{Vocab}(\overline{\kappa}, \phi_0) \cup \text{Vocab}(\overline{\kappa}, \phi_1) & \phi \text{ is } \phi_0 \rightarrow \phi_1 \end{cases}$$

It is extended to sets of formulas as follows:

$$\text{Vocab}(\overline{\kappa}, \text{T}) = \bigcup_{\phi \in \text{T}} \text{Vocab}(\overline{\kappa}, \phi).$$

Note that it is only in the propositional case that we can carry out this *static* analysis of the vocabulary of a sentence. This will not be possible in the quantified versions. Also note that our definition of vocabulary of a sentence is somewhat different from Guha's notion of definedness. Guha proposes to treat $\texttt{ist}(\kappa, \phi)$ as false if ϕ is not in the vocabulary of the context κ.

Satisfaction We can think of partial truth assignments as total truth assignments in a three-valued logic. Our satisfaction relation then corresponds to Bochvar's three valued logic, [2], since an implication is meaningless if either the antecedent or the consequent are meaningless. We chose Bochvar's three valued logic because we intend meaningfulness to be interpreted as syntactic meaningfulness, rather then semantic meaningfulness which could be ascribed to Kleene's three valued logic.

Definition ($\models$)**:** If $\nu \in \overline{\kappa}^{\mathfrak{M}}$ and Vocab$(\overline{\kappa}, \chi) \subseteq$ *Vocab*($\mathfrak{M}$), then we define satisfaction, $\mathfrak{M}, \nu \models_{\overline{\kappa}} \chi$, inductively on the structure of χ as follows:

$\mathfrak{M}, \nu \models_{\overline{\kappa}} \rho$ iff $\nu(\rho) = 1, \quad \rho \in \mathbb{P}$
$\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\phi$ iff not $\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi$
$\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi \rightarrow \psi$ iff $\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi$ implies $\mathfrak{M}, \nu \models_{\overline{\kappa}} \psi$
$\mathfrak{M}, \nu \models_{\overline{\kappa}} \texttt{ist}(\kappa_1, \phi)$ iff $\forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}} \quad \mathfrak{M}, \nu_1 \models_{\overline{\kappa} * \kappa_1} \phi$

If the preconditions $\nu \in \overline{\kappa}^{\mathfrak{M}}$ and Vocab$(\overline{\kappa}, \chi) \subseteq$ *Vocab*($\mathfrak{M}$) do not hold, then neither $\mathfrak{M}, \nu \models_{\overline{\kappa}} \chi$ nor $\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\chi$.

In the $\texttt{ist}$ clause of the satisfaction relation note that $\overline{\kappa} * \kappa_1 \in$ Dom($\mathfrak{M}$) since Vocab$(\overline{\kappa}, \texttt{ist}(\kappa_1, \phi)) \subseteq$ *Vocab*($\mathfrak{M}$), and the Dom($\mathfrak{M}$) is a rooted subtree; i.e. if $\overline{\kappa} > \overline{\kappa}_0$, then not $\mathfrak{M}, \nu \models_{\overline{\kappa}} \chi$. We write $\mathfrak{M} \models_{\overline{\kappa}} \chi$ iff (Vocab$(\overline{\kappa}, \chi) \subseteq$ *Vocab*($\mathfrak{M}$) and $\forall \nu \in \overline{\kappa}^{\mathfrak{M}} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \chi$); we also write $\models_{\overline{\kappa}} \chi$ iff for all models $\mathfrak{M}$ classical on Vocab $\quad \mathfrak{M} \models_{\overline{\kappa}} \chi$.

3 Decidability

The purpose of this section is to show that the propositional logic of contexts is decidable (i.e. that there is an effective procedure that says whether or not a given formula is valid, and hence also a theorem of the system). This will be done by showing that the propositional logic of contexts has the finite model property: any formula that is satisfiable is satisfiable in a model with finitely many finite truth assignments.

Definition (restriction of $\mathfrak{M}(\overline{\kappa})$): We first define the restriction of a single truth assignment, ν, with respect to $\mathrm{Vocab}(\overline{\kappa}_0, \phi)$ to be a truth assignment which, on the atoms that are in the scope of the context sequence $\overline{\kappa}$, corresponds to ν, and is false elsewhere.

$$\nu_{\mathrm{Vocab}(\overline{\kappa}_0,\phi),\overline{\kappa}} = \text{ the unique } \nu' \text{ such that } \nu'(p) = \begin{cases} \nu(p) & <\overline{\kappa}, p> \in \mathrm{Vocab}(\overline{\kappa}_0, \phi) \\ 0 & <\overline{\kappa}, p> \notin \mathrm{Vocab}(\overline{\kappa}_0, \phi) \end{cases}$$

The restriction of a set of truth assignments, V, with respect to $\mathrm{Vocab}(\overline{\kappa}_0, \phi)$ is defined as follows.

$$V_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} = \{\nu_{\mathrm{Vocab}(\overline{\kappa}_0,\phi),\overline{\kappa}} \mid \nu \in V\}.$$

Definition (restriction of $\mathfrak{M}$): The restriction of a model $\mathfrak{M}$ with respect to $\mathrm{Vocab}(\overline{\kappa}_0, \phi)$ is a model which maps every context sequence which appears in ϕ to the set of restricted truth assignments.

$$\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} : S \to \{\mathfrak{M}(\overline{\kappa})_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \mid \overline{\kappa} \in \mathrm{Dom}(\mathrm{Vocab}(\overline{\kappa}_0, \phi))\},$$

where S is the set of all subsequences of context sequences from $\mathrm{Dom}(\mathrm{Vocab}(\overline{\kappa}_0, \phi))$.

$$S = \{\overline{\kappa}_1 \mid \overline{\kappa}_1 \leq \overline{\kappa}_0 \quad \wedge \quad \exists \overline{\kappa}_2 \in \mathrm{Dom}(\mathrm{Vocab}(\overline{\kappa}_0, \phi)) \quad \overline{\kappa}_2 \leq \overline{\kappa}_1\}.$$

Thus the model $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}$ maps a context sequence $\overline{\kappa}$ to a set of truth assignments $\mathfrak{M}(\overline{\kappa})_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}$:

$$\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} : \overline{\kappa} \mapsto \mathfrak{M}(\overline{\kappa})_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}.$$

Theorem (finite model property): $\mathfrak{M} \models_{\overline{\kappa}_0} \phi$ iff $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}_0} \phi$.

Note that only the $(\Rightarrow)$ direction will be needed to prove the decidability of the propositional logic of context. We actually prove a stronger property:

$$\text{if} \quad \mathrm{Vocab}(\overline{\kappa}, \alpha) \subseteq \mathrm{Vocab}(\overline{\kappa}_0, \phi) \quad \text{then} \quad (\mathfrak{M} \models_{\overline{\kappa}} \alpha \quad \text{iff} \quad \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}} \alpha)$$

by induction on the structure of the formula ϕ.

Corollary (decidability): There is an effective procedure which will determine whether or not a formula given in some context is valid.

4 Comparison to Kripke Semantics

In this section we study the relationship between the semantics of context (as given in this paper) and Kripke semantics. We show that if all the aspects of partiality in the definition of a context model are disregarded, then most context models can be matched up to a particular class of Kripke models. Since there is always some leeway in the match, based on how we define the Kripke model and what notion of equivalence between context and Kripke models is taken, the models will not be matched precisely. We will discuss the adequacy of the match in more detail later.

We proceed to define some preliminaries for our construction.

Definition (non-partial model): A non-partial context model $\mathfrak{M}$ is a function which maps every context sequence $\overline{\kappa} \in \mathbb{K}^*$ to a set of total truth assignments,

$$\mathfrak{M} \in (\mathbb{K}^* \to \mathbf{P}(\mathbb{P} \to 2)).$$

Note that the two additional side conditions in the definition of the model are no longer needed. Also note that for the non-partial models the following property of the satisfaction relation holds:

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\phi \quad \text{iff} \qquad \text{not} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \phi.$$

Since in the comparison to Kripke semantics we will only be concerned with non-partial models, henceforth in this section we will refer to a partial context model simply as a context model. In this section we will also use the term "context logic" to refer to the general system described in §2 with the semantics restricted to non-partial models.

We now give a brief sketch of a standard propositional modal logic. We will be using a propositional modal logic with a countable number of modalities and its corresponding Kripke semantics. Given a context language specified by a possibly finite set of contexts, $\mathbb{K}$, and a set of propositional atoms, $\mathbb{P}$, we define a modal language consisting of the propositional atoms, $\mathbb{P}$, standard propositional connectives, $\neg$ and $\to$, and modalities, $\Box_1, \Box_2, \ldots$; one for each context from $\mathbb{K} = \{\kappa_\beta\}_{\beta<\alpha}$. We also define a bijective translation function which to each formula of the context logic, $\phi \in \mathbb{W}$, assigns a well-formed modal formula, $\phi^\Box$. The formula $\phi^\Box$ is obtained from ϕ by replacing each occurrence of $\mathtt{ist}(\kappa_\beta, \psi)$ in ϕ with $\Box_\beta(\psi^\Box)$. A *Kripke model* is a tuple $<\mathbb{S}, w_0, \pi, R_\beta>_{\beta<\alpha}$ where $\mathbb{S}$ is the set of *possible worlds*, $w_0 \in \mathbb{S}$ is the *actual world*, and π is a mapping from the the worlds in $\mathbb{S}$ to truth assignments over atomic propositions in $\mathbb{P}$, for some $\alpha \leq \omega$. Every R_β is a binary relation on $\mathbb{S}$. In order to distinguish Kripke models from context models, we use $\mathfrak{M}^\Box$ to refer to Kripke models. Kripke models are often called Kripke structures or possible-world structures. *Satisfaction* is defined to be a relation on a Kripke model, a world from that model, and a formula; it is written as $\mathfrak{M}^\Box, w \models \phi$. Note that the same symbol is used for satisfaction in the context logic, however it will be obvious from the arguments of the relation which satisfaction relation is being referred to. Atomic formulas are satisfied at a world if they are made true by the truth assignment associated with that world. Satisfaction for propositional connectives is defined as in classical propositional logic. The formula $\Box_\beta\, \phi$ is satisfied at a world w iff ϕ is satisfied at every world w' s.t. $w\, R_\beta\, w'$.

In order to compare a context model and a Kripke model, we need to know which worlds are intended to describe the same state of affairs as a given context, i.e. which worlds are associated to which context.

Definition (association relation): A relation A is an association relation from the context model $\mathfrak{M}$ to the Kripke model $\mathfrak{M}^\Box = <\mathbb{S}, w_0, \pi, R_\beta>_{\beta<\alpha}$ iff

1. $A \subseteq \mathbb{K}^* \times (\mathbb{P} \to 2) \times \mathbb{S}$
2. $(\forall \overline{\kappa})(\forall \nu \in \mathfrak{M}(\overline{\kappa}))(\exists w) \quad A(\overline{\kappa}, \nu, w)$
3. $A(\overline{\kappa}, \nu, w)$ implies $(\forall w')(\exists \beta)$ $(w R_\beta w'$ implies $(\exists \kappa')(\exists \nu')$ $A(\overline{\kappa} * \kappa', \nu', w'))$
4. $A(\overline{\kappa}, \nu, w)$ implies $\pi(w) = \nu$
5. $A(\epsilon, \pi(w_0), w_0)$

Given a context sequence $\overline{\kappa}$ and a truth assignment from $\mathfrak{M}(\overline{\kappa})$, the association relation expresses which world is to be associated with the truth assignment in that context, and vice versa. Note that the same truth assignment in different context sequences may produce different worlds.

In order to be able to compare a context model and a Kripke model, we need a notion of what it means for the two models to be equivalent.

Definition (elementary equivalence): A context model $\mathfrak{M}$ is elementarily equivalent to a Kripke model $\mathfrak{M}^\Box$ with respect to association A $(\mathfrak{M} \equiv \mathfrak{M}^\Box$ w.r.t. $A)$ iff

$$A(\overline{\kappa}, \nu, w) \quad \text{implies} \quad (\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi \quad \text{iff} \quad \mathfrak{M}^\Box, w \models \phi^\Box)$$

for any context sequence $\overline{\kappa}$, truth assignment $\nu \in \mathfrak{M}(\overline{\kappa})$, world $w \in \mathbb{S}$, and wff ϕ.

We introduce a class of context models, called the *actual models*. They have the property that the empty context sequence is associated with a single truth assignment, which is interpreted as the actual state of affairs or the state of affairs in the actual world.

Definition ($\mathfrak{Actual}$)**:** A context model $\mathfrak{M}$ is an *actual model*, $\mathfrak{M} \in \mathfrak{Actual}$, iff $|\mathfrak{M}(\epsilon)| = 1$.

Theorem (representation1): For every context model $\mathfrak{M} \in \mathfrak{Actual}$, there exists a Kripke model $\mathfrak{M}^\Box$, and an association A from $\mathfrak{M}$ to $\mathfrak{M}^\Box$ such that $\mathfrak{M} \equiv \mathfrak{M}^\Box$ w.r.t. A.

In the proof, given a context model $\mathfrak{M}$, we construct an elementarily equivalent Kripke model $\mathfrak{M}^\Box$. Intuitively, for every truth assignment in every context sequence, we create a world in the Kripke model. The function π of a world is the same as the truth assignment that created that world. We then define relations on these worlds and show that the created structure, $\mathfrak{M}^{\Box}$, is in fact a Kripke model. Finally, we prove that the two models are in fact elementarily equivalent.

We will not be able to represent all Kripke models using context models. To define the class of Kripke models which can be represented using context models, we introduce a new property of accessibility relations of Kripke structures.

Definition ($\Phi_1(R_i, R_j)$): The ordered pair of relations $<R_i, R_j>$ satisfies the property Φ_1, formally $\Phi_1(R_i, R_j)$, iff

$$(\forall w_1 \forall w_2 \forall w_2' \forall w_3)\ ((w_1 R_i w_2 \text{ and } w_2 R_j w_3 \text{ and } w_1 R_i w_2') \text{ implies } w_2' R_j w_3).$$

Definition (C_1): Let C_1 be the classes of Kripke models in which all the ordered pairs of accessibility relations satisfy the property Φ_1.

Theorem (representation2): For every Kripke model $\mathfrak{M}^{\Box} \in \mathsf{C}_1$ there exists a context model $\mathfrak{M}$ such that $\mathfrak{M} \equiv \mathfrak{M}^{\Box}$ w.r.t. A.

In the proof, given a Kripke model $\mathfrak{M}^{\Box}$, we construct an elementarily equivalent context model $\mathfrak{M}$. Intuitively, we will identify context sequences with paths through the Kripke model $\mathfrak{M}^{\Box}$. Then, the truth assignments of the worlds in $\mathfrak{M}^{\Box}$ which can be reached via a path $\overline{\kappa}$ will be placed in $\mathfrak{M}(\overline{\kappa})$. Finally, we prove that the two models are in fact elementarily equivalent.

4.1 Discussion

The representation results in this section depend on our definition of the Kripke model and the definition of elementary equivalence. Variations in either of these definitions would slightly change the theorems. For example, sometimes the actual world is excluded from the definition of the Kripke model. In this case we could generalize the (**representation1 theorem**) to hold for any context model, rather than only those in $\mathfrak{Actual}$. For (**representation2 theorem**) to hold we would need a way of including multiple subtrees in a single context model. One solution would be to connect all the subtrees to ϵ and associate the empty vocabulary with the empty sequence. To take another example, stronger results could be obtained by insisting that the association relation matches every world to some truth assignment in some context sequence. But here again, to prove (**representation2 theorem**) we would need vocabularies or a stronger notion of a context model which would allow multiple rooted subtrees as the domain of $\mathfrak{M}$. To conclude, there is always some leeway in representation results, based on the basic definitions. The main purpose of this section is to give the general flavor of the relations in expressiveness of the two kinds of models and a methodology which can be used for comparing context models and Kripke models.

5 Conclusions and Future Work

Our goal is to extend the system to a full quantification logic. One advantage of quantificational system is that it enables us to express relations between context, operations on contexts, and state *lifting rules* which describe how a fact from one context can be used in another context. However, in the presence of context variables it might not be possible to define the vocabulary of a sentence without knowing which object a variable is bound to. Therefore, the first step in this direction is to to examine propositional systems with dynamic definitions of meaningfulness.

We also plan to define non-Hilbert style formal systems for context. Probably the most relevant is a natural deduction system, which would be in line with McCarthy's original proposal of treating contextual reasoning as a strong version of natural deduction. In such a system, entering a context would correspond to making an assumption in natural deduction, while exiting a context corresponds to discharging an assumption.

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