METAMATHEMATICS OF CONTEXTS

Saša Buvač
Computer Science Department
Stanford University
Stanford
California 94305-2140, USA
buvac@sail.stanford.edu

Vanja Buvač
HB 455
Dartmouth College
Hanover
New Hampshire 03755, USA
vanja@dartmouth.edu

Ian A. Mason
Computer Science Department
Stanford University
Stanford
California 94305-2140, USA
iam@sail.stanford.edu

Abstract

In this paper we investigate the simple logical properties of *contexts*. We describe both the syntax and semantics of a general propositional language of context, and we give a Hilbert style proof system for this language. A propositional logic of context extends classical propositional logic in two ways. Firstly, a new modality, $\mathtt{ist}(\kappa, \phi)$, is introduced. It is used to express that the sentence, ϕ, holds in the context, κ. Secondly, each context has its own vocabulary, i.e. a set of propositional atoms which are *defined* or *meaningful* in that context. The main results of this paper are the soundness and completeness of this Hilbert style proof system. We also provide soundness and completeness results (i.e., correspondence theory) for various extensions of the general system. Finally, we prove that our logic is decidable, and give a brief comparison of our semantics to Kripke semantics.

1 Introduction

In this paper we investigate the simple logical properties of *contexts*. Contexts were first introduced into AI by John McCarthy in his Turing Award Lecture, [16], as an approach that might lead to the solution of the problem of *generality* in AI. This problem is simply that existing AI systems lack generality.

Since then, contexts have found many uses in various areas of AI. R. V. Guha's doctoral dissertation, [12] , under McCarthy's supervision was the first in-depth study of context. Guha's context research was primarily motivated by the Cyc system, [13], a large common-sense knowledge-base currently being developed at MCC. Without contexts it would have been virtually impossible to create and successfully manage a knowledge base of the size of Cyc.

The knowledge sharing community has also accepted the need for explicating context. Their concern with transferring information from one agent to another can be declaratively expressed using the context formalism. Currently, proposals for introducing contexts into the Knowledge Interchange Format (KIF), [9], are being considered. Furthermore, it seems that the context formalism can provide semantics for the process of translating facts into KIF and from KIF, which is one of the key tasks that the knowledge sharing effort is facing.

Furthermore, computational linguists use contexts to account for the phenomena that the meaning of an English sentence depends on the context in which it is uttered. They have identified various properties and characterizations of such contexts. For example, Barbara Grosz in her Ph.D. thesis, [11], implicitly captures the context of a discourse by "focusing" on the objects and actions which are most relevant to the discourse. This notion is similar to an ATMS context, [8], which is simply a list of propositions that are "assumed" by the reasoning system.

However, until now no formal logical explication of contexts has been given. The aim of this paper is to rectify this deficiency. We describe both the syntax and semantics of a general propositional language of context, and give a Hilbert style proof system for this language. The main results of this paper are the soundness and completeness of this Hilbert style proof system. We also provide soundness and completeness results (i.e., correspondence theory) for various extensions of the general system. Finally, we show that our logic is decidable.

This paper is organized as follows. §1, this one, serves as an introduction to both the paper and our notation. In §2 we describe the syntax, semantics and proof theory of the general system. We also establish the soundness and completeness of this proof system. In §3 we provide correspondence results for several variations on the general system. §4 is dedicated to the completeness of a slightly more elaborate variation on the general system, namely the propositional fragment of Guha's logic of context, [12]. In §5 we prove that our logic is decidable. A comparison of our semantics to Kripke semantics is given in §6. §7 describes related work, and §8 contains our conclusions. Most of the results in §2 through §4 were first announced in [5]; the result in §5 was first announced in [6]; most of the results in §6 were first announced in [4].

1.1 Motivation

Our main motivation for formalizing contexts is to solve the problem of generality in AI. We want to be able to make AI systems which are never permanently stuck with the concepts they use at a given time because they can always transcend the context they are in. Such a capability would allow the designer of a reasoning system to include only such phenomena as are required for the system's immediate purpose, retaining the assurance that if a broader system is required later, "lifting axioms" can be devised to restate the facts from the narrow context in the broader context with qualifications added as necessary. We provide two simple examples.

The first example is due to McCarthy [18]. It illustrates how a reasoning system can utilize contexts to incorporate information from a general common sense knowledge base into other specialized knowledge bases. Assume that in the context of situation calculus $on(x, y, s)$ is used to express the fact that object x is on top of object y in situation s. Although no mention to the notion of *above* is made in the context of situation calculus, we are interested to know which of the *above* relations hold in a particular situation. The definition of *above* in terms of *on* is likely to be found in some context of general common sense knowledge. The context formalism will allow a reasoning system to use the theory of situation calculus and the theory of general common sense knowledge together. Furthermore, in the logic we can write axioms to import or *lift* the definition of *above* from the context of general common sense knowledge into the context of situation calculus. Although *above* was not originally defined in the context of situation calculus, the system, after lifting, will be able to infer which *above* relations hold in a particular situation. Of course, the power of a full quantificational logic will be needed to adequately address this example.

The second example concerns theories which were not originally intended to be used together, and in fact might, on the surface, seem inconsistent. For example, assume a common sense knowledge base of Stanford University contains the proposition "kids drive BMW's". A common sense knowledge base of Berkeley, which was not originally intended to be used with the above mentioned Stanford knowledge base, will probably contain the negation of this proposition. A logic of context will enable a reasoning system to use such seemingly inconsistent knowledge bases without deriving a contradiction.

1.2 Notation

We use standard mathematical notation. If X and Y are sets, then $X \rightarrow_{\mathrm{p}} Y$ is the set of partial functions from X to Y. $\mathbf{P}(X)$ is the set of subsets of X. X^* is the set of all finite sequences, and we let $\bar{x} = [x_1, \ldots, x_n]$ range over X^*. ϵ is the empty sequence. We use the infix operator $*$ for appending sequences. We make no distinction between an element and the singleton sequence containing that element. Thus we write $\bar{x} * x_1$ instead of $\bar{x} * [x_1]$. As is usual in logic we treat X^* as a tree (that grows downward). $\bar{x}_1 < \bar{x}_0 \leq \epsilon$ iff $\bar{x}_1$ properly extends $\bar{x}_0$ (i.e. $(\exists \bar{y} \in X^* - \{\epsilon\})(\bar{x}_1 = \bar{x}_0 * \bar{y})$). We say $Y \subseteq X^*$ is a subtree rooted at $\bar{y}$ to mean

1. $\bar{y} \in Y$ and $(\forall \bar{z} \in Y)(\bar{z} \leq \bar{y})$
2. $(\forall \bar{z} \in Y)(\forall \bar{w} \in X^*)(\bar{z} \leq \bar{w} \leq \bar{y} \rightarrow \bar{w} \in Y)$

2 The General System

A propositional logic of context extends classical propositional logic in two ways. Firstly, a new modality, $\mathtt{ist}(\kappa, \phi)$, is introduced. It is used to express that the sentence, ϕ, holds in the context, κ. Secondly, each context has its own vocabulary, i.e. a set of propositional atoms which are *defined* or *meaningful* in that context. The vocabulary of one context may or may not overlap with another context.

2.1 Syntax

We begin with two distinct countably infinite sets, $\mathbb{K}$ the set of all contexts, and $\mathbb{P}$ the set of propositional atoms. The set, $\mathbb{W}$, of well-formed formulas (wffs) is built up from the propositional atoms, $\mathbb{P}$, using the usual propositional connectives (negation and implication) together with the ist modality.

Definition ($\mathbb{W}$): $\mathbb{W} = \mathbb{P} \cup (\neg\mathbb{W}) \cup (\mathbb{W} \rightarrow \mathbb{W}) \cup \mathtt{ist}(\mathbb{K}, \mathbb{W})$

The operations $\wedge$, $\vee$ and $\leftrightarrow$ are defined as abbreviations in the usual way. The term *literal* is used to refer to a propositional atom or the negation of a propositional atom. We use $\pm\phi$ to represent either the formula ϕ, or its negation $\neg\phi$. We also use the following abbreviations:

$$
\begin{aligned}
\mathtt{ist}(\epsilon, \phi) &:= \phi \\
\mathtt{ist}(\overline{\kappa}, \phi) &:= \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \ldots, \mathtt{ist}(\kappa_n, \phi))) \\
\mathtt{ist}^{\pm}(\overline{\kappa}, \phi) &:= \pm\mathtt{ist}(\kappa_1, \pm\mathtt{ist}(\kappa_2, \cdots \pm \mathtt{ist}(\kappa_n, \phi) \cdots))
\end{aligned}
$$

when $\overline{\kappa}$ is the context sequence $[\kappa_1, \kappa_2, \ldots, \kappa_n]$. In the definition of $\mathtt{ist}^{\pm}$ all the ist's need not be of the same parity. PROP is the set of all well formed formulas which do not contain ist's. If ψ is a formula containing distinct atoms $p_1, \ldots, p_n$, then we write $\psi(\phi_1, \ldots, \phi_n)$ for the formula which results from ψ by simultaneously replacing all the occurrences of p_i in ψ by ϕ_i. We say that $\psi(\phi_1, \ldots, \phi_n)$ is an *instance* of ψ.

We have chosen to develop a modal logic rather than to reifying sentences and treat ist as a regular predicate, because (1) it leads to a more natural semantics (defined in the following subsection), and (2) the language does not allow self-referential statements, thus avoiding paradoxes. Although self-referential formulas are relevant for developing theories of truth, they are not needed for describing states of affairs which hold in particular contexts. Therefore, the loss of expressive power due to lack of self-referential formulas will not be missed in our logic. The latter approach, of reifying formulas, is taken by Attardi and Simi in [1]. We further discuss their work in §7.

2.2 Semantics

We begin with a system which makes as few semantic restrictions as possible. Other systems, defined in section §3, are obtained by placing restrictions on the models. To explain the semantics we first introduces a naïve notion of a model, which is then refined in two stages.

Naïvely, a context is modelled by a set of truth assignments, that describe the possible states of affairs of that context. Thus a model will associate a set of truth assignments with every context. These truth assignments reflect the states of affairs which are possible in a context. For a proposition to be true in a context it has to be satisfied by all the truth assignments associated with that context. Therefore, the ist modality is interpreted as validity: $\mathtt{ist}(\kappa, \rho)$ is true iff the propositional atom ρ is true in all the truth assignments associated with context κ. Treatment of ist as validity corresponds to Guha's proposal for context semantics, which was motivated by the Cyc knowledge base. A system which models a context by a single truth assignment, thus interpreting `ist` as truth, can be obtained by placing simple restrictions on the definition of a model and by enriching the set of axioms.

However, this naïve model is not powerful enough to represent some properties desired of contexts. Therefore, we need to refine our naïve notion of a model. We do this in two stages:

Firstly, the nature of a particular context may itself be context dependent. For example, in the context of the 1950's, the context of car racing is different from the context of car racing viewed from today's context. This naturally leads to considering sequences of contexts rather than a context in isolation. So, a model will associate a set of truth assignments with a context sequence, rather than an individual context (as was the case in the naïve view). We refer to this feature of the system as *non-flatness.* It reflects on the intuition that what holds in a context can depend on how this context has been reached, i.e. from which perspective it is being viewed. For example, non-flatness will be desirable if we represent the beliefs of an agent as the sentences which hold in a context. A system of flat contexts can easily be obtained by placing certain restrictions on what kinds of structures are allowed as models, as well as enriching the axiom system (cf. §3.2).

Secondly, since different contexts can have different vocabularies, some propositions can be meaningless in some contexts, and therefore the truth assignments describing the state of affairs in that context need to be partial.

Now we are ready to define the general model:

Definition ($\mathfrak{M}$): In this system a model, $\mathfrak{M}$, will be a function which maps a context sequence $\overline{\kappa} \in \mathbb{K}^*$ to a set of partial truth assignments,

$$\mathfrak{M} \in (\mathbb{K}^* \rightarrow_{\mathrm{p}} \mathbf{P}(\mathbb{P} \rightarrow_{\mathrm{p}} 2)),$$

with the added conditions that

1. $(\forall \overline{\kappa})(\forall \nu_1, \nu_2 \in \mathfrak{M}(\overline{\kappa}))(\mathrm{Dom}(\nu_1) = \mathrm{Dom}(\nu_2))$
2. $\mathrm{Dom}(\mathfrak{M})$ is a subtree of $\mathbb{K}^*$ rooted at some context sequence $\overline{\kappa}_0$.

We write $\overline{\kappa}^{\mathfrak{M}}$ to denote the set of partial truth assignments $\mathfrak{M}(\overline{\kappa})$. Note that $\overline{\kappa}^{\mathfrak{M}}$ can be empty. Since all the elements of $\mathfrak{M}(\overline{\kappa})$ have the same domain, which is imposed by condition 1. above, we will write $\mathrm{Dom}(\mathfrak{M}(\overline{\kappa}))$ to refer to this domain. The collection of all such models will be denoted by $\mathbb{M}$.

Note that condition 1 is actually not a restriction, in the following sense. Given a model $\mathfrak{M}$ we construct a new model (satisfying 1.) by restricting each assignment (associated to a context) to the atoms which all assignments (associated to that context) have in common. The resulting model will satisfy the same sentences. Formally, assume $\mathfrak{M}$ is a model which does not satisfy restriction 1. We can define a model $\mathfrak{M}'$ which satisfies condition 1, in the following fashion:

$$\mathrm{Dom}(\mathfrak{M}'(\overline{\kappa})) := \bigcap_{\nu \in \mathfrak{M}(\overline{\kappa})} \mathrm{Dom}(\nu),$$

$$\mathfrak{M}'(\overline{\kappa}) := \{\nu' \mid (\exists \nu \in \mathfrak{M}(\overline{\kappa}))(\nu' \sqsubseteq \nu) \quad \text{and} \quad \mathrm{Dom}(\nu') = \mathrm{Dom}(\mathfrak{M}'(\overline{\kappa}))\}.$$

It will turn out that models $\mathfrak{M}$ and $\mathfrak{M}'$ satisfy the same formulas.

We could have assumed the existence of a *fixed outermost context* which would result in $\mathrm{Dom}(\mathfrak{M})$ being a tree rooted at empty sequence ϵ (i.e. the fixed outermost context). This would result in slightly simpler notation and proofs. However, although more complicated, our definition is based on the intuition that there is no *outermost* context. Since there is no outermost context, it should be possible to write automated reasoning systems which are never permanently stuck with the concepts they use at a given time because they can always transcend their current context.

2.2.1 Vocabularies

To capture the intuition that different contexts can have different vocabularies, we make the truth assignments in our model partial. The atoms which are given a truth value in a context sequence are defined by a relation $\mathsf{Vocab} \subseteq \mathbb{K}^* \times \mathbb{P}$. Given a Vocab, the *vocabulary of a context sequence* $\overline{\kappa}$, or the set of atoms which are meaningful in that sequence, is $\{\rho \mid <\overline{\kappa}, \rho> \in \mathsf{Vocab}\}$.

Definition (*Vocab* **of** $\mathfrak{M}$)**:** We define a function $Vocab : \mathbb{M} \to \mathbf{P}(\mathbb{K}^* \times \mathbb{P})$, which given a model returns the vocabulary of the model:

$$Vocab(\mathfrak{M}) := \{<\overline{\kappa}, \rho> \mid \overline{\kappa} \in \mathrm{Dom}(\mathfrak{M}) \text{ and } \rho \in \mathrm{Dom}(\mathfrak{M}(\overline{\kappa}))\}$$

We say that a model $\mathfrak{M}$ is *classical on vocabulary* Vocab iff $\mathsf{Vocab} \subseteq Vocab(\mathfrak{M})$.

The notion of vocabulary can also be applied to sentences. Intuitively, the vocabulary of a sentence relates a context sequence to the atoms which occur in the scope of that context sequence. In the definition we also need to take into account that sentences are not given in isolation but in a context.

Definition (Vocab **of** ϕ **in** $\overline{\kappa}$)**:** We define a function $\mathrm{Vocab} : \mathbb{K}^* \times \mathbb{W} \to \mathbf{P}(\mathbb{K}^* \times \mathbb{P})$ which given formula in a context, returns the vocabulary of the formula.

$$\mathrm{Vocab}(\overline{\kappa}, \phi) = \begin{cases} \{<\overline{\kappa}, \phi>\} & \phi \in \mathbb{P} \\ \mathrm{Vocab}(\overline{\kappa}, \phi_0) & \phi \text{ is } \neg\phi_0 \\ \mathrm{Vocab}(\overline{\kappa} * \kappa, \phi_0) & \phi \text{ is } \mathtt{ist}(\kappa, \phi_0) \\ \mathrm{Vocab}(\overline{\kappa}, \phi_0) \cup \mathrm{Vocab}(\overline{\kappa}, \phi_1) & \phi \text{ is } \phi_0 \to \phi_1 \end{cases}$$

It is extended to sets of formulas as follows:

$$\mathrm{Vocab}(\overline{\kappa}, \mathrm{T}) = \bigcup_{\phi \in \mathrm{T}} \mathrm{Vocab}(\overline{\kappa}, \phi).$$

Note that it is only in the propositional case that we can carry out this *static* analysis of the vocabulary of a sentence. This will not be possible in the quantified versions. Also note that our definition of vocabulary of a sentence is somewhat different from Guha's notion of definedness. Guha proposes to treat $\mathtt{ist}(\kappa, \phi)$ as false if ϕ is not in the vocabulary of the context κ.

2.2.2 Satisfaction

We can think of partial truth assignments as total truth assignments in a three-valued logic. Our satisfaction relation then corresponds to Bochvar's three valued logic, [2], since an implication is meaningless if either the antecedent or the consequent are meaningless. We chose Bochvar's three valued logic because we intend meaningfulness to be interpreted as syntactic meaningfulness, rather then semantic meaningfulness which could be ascribed to Kleene's three valued logic.

Definition ($\models$): If $\nu \in \overline{\kappa}^{\mathfrak{M}}$ and $\mathrm{Vocab}(\overline{\kappa}, \chi) \subseteq \mathit{Vocab}(\mathfrak{M})$, then we define satisfaction, $\mathfrak{M}, \nu \models_{\overline{\kappa}} \chi$, inductively on the structure of χ as follows:

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \rho \text{ iff } \nu(\rho) = 1, \quad \rho \in \mathbb{P}$$

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\phi \text{ iff not } \mathfrak{M}, \nu \models_{\overline{\kappa}} \phi$$

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi \rightarrow \psi \text{ iff } \mathfrak{M}, \nu \models_{\overline{\kappa}} \phi \text{ implies } \mathfrak{M}, \nu \models_{\overline{\kappa}} \psi$$

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \text{ iff } \forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}} \quad \mathfrak{M}, \nu_1 \models_{\overline{\kappa} * \kappa_1} \phi$$

If the preconditions $\nu \in \overline{\kappa}^{\mathfrak{M}}$ and $\mathrm{Vocab}(\overline{\kappa}, \chi) \subseteq \mathit{Vocab}(\mathfrak{M})$ do not hold, then neither $\mathfrak{M}, \nu \models_{\overline{\kappa}} \chi$ nor $\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\chi$.

In the `ist` clause of the satisfaction relation note that $\overline{\kappa} * \kappa_1 \in \mathrm{Dom}(\mathfrak{M})$ since $\mathrm{Vocab}(\overline{\kappa}, \mathtt{ist}(\kappa_1, \phi)) \subseteq \mathit{Vocab}(\mathfrak{M})$, and the $\mathrm{Dom}(\mathfrak{M})$ is a rooted subtree; i.e. if $\overline{\kappa} > \overline{\kappa}_0$, then not $\mathfrak{M}, \nu \models_{\overline{\kappa}} \chi$. We write $\mathfrak{M} \models_{\overline{\kappa}} \chi$ iff ($\mathrm{Vocab}(\overline{\kappa}, \chi) \subseteq \mathit{Vocab}(\mathfrak{M})$ and $\forall \nu \in \overline{\kappa}^{\mathfrak{M}} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \chi$); we also write $\models_{\overline{\kappa}} \chi$ iff for all models $\mathfrak{M}$ classical on Vocab $\quad \mathfrak{M} \models_{\overline{\kappa}} \chi$.

2.3 Formal System

We now present the formal system. To do this we fix a particular vocabulary, $\mathsf{Vocab} \subset \mathbb{K}^* \times \mathbb{P}$, and define a provability relation, $\vdash_{\overline{\kappa}}^{\mathsf{Vocab}}$. Since Vocab will remain fixed throughout we omit explicitly mentioning it and write $\vdash_{\overline{\kappa}} \phi$ instead. Similarly, to avoid constantly stating lengthy side conditions we make the following convention.

Definedness Convention: *In the sequel, whenever we write $\vdash_{\overline{\kappa}} \phi$ we will be assuming implicitly that* $\mathrm{Vocab}(\overline{\kappa}, \phi) \subseteq \mathsf{Vocab}$.

Axioms and inference rules are given in table 1. Note that the rules of inference preserve the (**definedness convention**).

Assuming that our system was limited to only one context, the rule (**CS**) would be identical to the rule of necessitation in normal systems of modal logic, and the axiom schema (**K**) would be identical to the the standard axiom schema K. So, by ignoring axiom schema ($\triangle$), in the single context case, our formal system is identical to what is usually called the *normal system* of modal logic, characterized by (**PL**), (**MP**), (**K**), and the rule of necessitation. The axiom schema ($\triangle$) is needed in order to accommodate the validity aspect of the `ist` modality; it is derivable in the system which treats `ist` as truth (see §3.3) and does not allow inconsistent contexts. We will also discuss the ramifications of this schema in §6 and §7.

(PL)	$\vdash_{\overline{\kappa}} \phi$	provided ϕ is an instance of a tautology.	
(K)	$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi \to \psi) \to \mathtt{ist}(\kappa_1, \phi) \to \mathtt{ist}(\kappa_1, \psi)$		
($\triangle$)	$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \vee \psi) \to \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \vee \mathtt{ist}(\kappa_1, \psi)$		
(MP)	$\dfrac{\vdash_{\overline{\kappa}} \phi \quad \vdash_{\overline{\kappa}} \phi \to \psi}{\vdash_{\overline{\kappa}} \psi}$	(CS) $\dfrac{\vdash_{\overline{\kappa} * \kappa_1} \phi}{\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi)}$	provided $\overline{\kappa} \leq \overline{\kappa}_0$.

Table 1: Axioms and Inference Rules

2.3.1 Provability

A formula ϕ *is provable in context* $\overline{\kappa}$ *with vocabulary* **Vocab** (formally $\vdash_{\overline{\kappa}} \phi$) iff $\vdash_{\overline{\kappa}} \phi$ is an instance of an axiom schema or follows from provable formulas by one of the inference rules; formally, iff there is a sequence $[\vdash_{\overline{\kappa}_1} \phi_1, \ldots, \vdash_{\overline{\kappa}_n} \phi_n]$ such that $\overline{\kappa}_n = \overline{\kappa}$, and $\phi_n = \phi$ and for each $i \leq n$ either $\vdash_{\overline{\kappa}_i} \phi_i$ is an axiom, or is derivable from the earlier elements of the sequence via one of the inference rules. In the case of assumptions, formula ϕ is provable from assumptions T in context $\overline{\kappa}_0$ with vocabulary **Vocab** (formally $\mathrm{T} \vdash^{\mathsf{Vocab}}_{\overline{\kappa}_0}$, or again taking into account that **Vocab** is fixed $\mathrm{T} \vdash_{\overline{\kappa}_0} \phi$) iff there are formulas $\phi_1, \ldots, \phi_n \in \mathrm{T}$, such that $\vdash_{\overline{\kappa}_0} (\phi_1 \wedge \cdots \wedge \phi_n) \to \phi$. Note that due to the definedness convention if $\mathrm{T} \vdash_{\overline{\kappa}_0} \phi$ then $\mathrm{Vocab(T)} \subseteq$ **Vocab**.

2.4 Consequences

We now prove the following useful theorems and derivable rules of this system.

(C) $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \wedge \mathtt{ist}(\kappa_1, \psi) \to \mathtt{ist}(\kappa_1, \phi \wedge \psi)$

(Or) $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \vee \mathtt{ist}(\kappa_1, \psi) \to \mathtt{ist}(\kappa_1, \phi \vee \psi)$

(M) $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi \wedge \psi) \to \mathtt{ist}(\kappa_1, \phi) \wedge \mathtt{ist}(\kappa_1, \psi)$

(ND) $\vdash_{\overline{\kappa}} \neg\mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \to \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa_2, \phi))$

($\triangle_-$) $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa_2, \phi) \vee \psi) \to \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa_2, \phi)) \vee \mathtt{ist}(\kappa_1, \psi)$

(CSE) $\dfrac{\vdash_{\overline{\kappa} * \kappa_1} \phi \leftrightarrow \psi}{\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \leftrightarrow \mathtt{ist}(\kappa_1, \psi)}$

(REP) $\dfrac{\vdash_{\overline{\kappa}} \phi_1 \leftrightarrow \phi_1' \cdots \vdash_{\overline{\kappa}} \phi_n \leftrightarrow \phi_n'}{\vdash_{\overline{\kappa}} \psi(\phi_1, \ldots, \phi_n) \leftrightarrow \psi(\phi_1', \ldots, \phi_n')}$ provided $\psi(p_1, \ldots p_n) \in \mathrm{PROP}$.

where $p_1, \ldots p_n$ are some new distinct propositional atoms. (**M**) is easily derivable from (**K**), (**PL**), and (**MP**). The proofs of (**C**), (**Or**), and (**REP**) are identical to proofs of corresponding theorems in a normal system of modal logic [7]. Note that (**M**) is the converse of (**C**).

We first observe that not only are (**ND**) and ($\triangle_-$) derivable in our system, but that both (**ND**) and ($\triangle_-$) are equivalent to ($\triangle$). The proof follows.

Proof (ND): We start with an instance of a propositional tautology:

$$\vdash_{\overline{\kappa}*\kappa_1} \mathtt{ist}(\kappa_2, \phi) \vee \neg \mathtt{ist}(\kappa_2, \phi).$$

Now by the context switching rule (**CS**) we obtain

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \vee \neg \mathtt{ist}(\kappa_2, \phi)).$$

Using ($\triangle$) and (**MP**):

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \vee \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi)).$$

Which by definition of disjunction is just

$$\vdash_{\overline{\kappa}} \neg \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi)).$$

This completes the proof of (**ND**) from ($\triangle$). Now we show that in our system ($\triangle$) is derivable from (**ND**). We start with an instance of (**ND**)

$$\vdash_{\overline{\kappa}} \neg \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi)).$$

By propositional logic it follows that

$$\vdash_{\overline{\kappa}} (\mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \psi)) \rightarrow (\neg \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \psi)).$$

Together with an instance of axiom schema (**K**):

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi) \rightarrow \psi) \rightarrow (\mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \psi))$$

using (**PL**) and (**MP**) this allows us to infer

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi) \rightarrow \psi) \rightarrow (\neg \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \psi))$$

which is in fact ($\triangle$). $\square_{\mathrm{ND}}$

The proof of (**ND**) from ($\triangle_-$) is identical to the proof of (**ND**) from ($\triangle$) with the exception that ($\triangle_-$) is used in the place where ($\triangle$) was used before. Now all that remains to be shown is that ($\triangle_-$) is derivable from (**ND**) in our system.

Proof ($\triangle_-$): We start with an instance of (**ND**)

$$\vdash_{\overline{\kappa}} \neg \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)).$$

By propositional logic it follows that

$$\vdash_{\overline{\kappa}} (\mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \psi)) \rightarrow (\neg \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \psi)).$$

Together with an instance of axiom schema (**K**):

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \rightarrow \psi) \rightarrow (\mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \psi))$$

using (**PL**) and (**MP**) this allows us to infer

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \rightarrow \psi) \rightarrow (\neg \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa_2, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \psi))$$

which is in fact $(\triangle_-)$. $\Box_{\triangle_-}$

Proof (CSE): Assume $\vdash_{\overline{\kappa}*\kappa_1} \phi \to \psi$. Applying the context switching rule (**CS**) we get $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi \to \psi)$, and using the axiom schema (**K**) it follows that $\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \to \mathtt{ist}(\kappa_1, \psi)$. The other direction is identical. $\Box_{\mathbf{CSE}}$

Note that in the same way that the context switching rule is analogous to the rule of necessitation (**RN**) in modal logic, the (**CSE**) rule resembles the (**RE**) rule in modal logic (from $\phi \leftrightarrow \psi$ infer $\Box\phi \leftrightarrow \Box\psi$).

The context principles are easily generalized to utilize the abbreviation $\mathtt{ist}(\overline{c}, \phi)$, resulting in theorems:

(K*) $\vdash_{\overline{\kappa}} \mathtt{ist}(\overline{c}, \phi \to \psi) \to \mathtt{ist}(\overline{c}, \phi) \to \mathtt{ist}(\overline{c}, \psi)$

$(\triangle^*)$ $\vdash_{\overline{\kappa}} \mathtt{ist}(\overline{c}, \mathtt{ist}(\kappa_1, \phi) \vee \psi) \to \mathtt{ist}(\overline{c}, \mathtt{ist}(\kappa_1, \phi)) \vee \mathtt{ist}(\overline{c}, \psi)$

$(\triangle^*_-)$ $\vdash_{\overline{\kappa}} \mathtt{ist}(\overline{c}, \neg\mathtt{ist}(\kappa_1, \phi) \vee \psi) \to \mathtt{ist}(\overline{c}, \neg\mathtt{ist}(\kappa_1, \phi)) \vee \mathtt{ist}(\overline{c}, \psi)$

(CS*) $\dfrac{\vdash_{\overline{\kappa}*\overline{c}} \phi}{\vdash_{\overline{\kappa}} \mathtt{ist}(\overline{c}, \phi)}$

Proof (K*): The proof by induction on the length of the context sequence $\overline{c}$. The base case is the schema (**K**). To prove the inductive step we start with the inductive hypothesis:

$$\vdash_{\overline{\kappa}*\kappa} \mathtt{ist}(\overline{c}, \phi \to \psi) \to \mathtt{ist}(\overline{c}, \phi) \to \mathtt{ist}(\overline{c}, \psi).$$

Applying the context switching rule (**CS**) we get:

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa, \mathtt{ist}(\overline{c}, \phi \to \psi) \to \mathtt{ist}(\overline{c}, \phi) \to \mathtt{ist}(\overline{c}, \psi))$$

and together with schema (**K**) this gives:

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa * \overline{c}, \phi \to \psi) \to \mathtt{ist}(\kappa, \mathtt{ist}(\overline{c}, \phi) \to \mathtt{ist}(\overline{c}, \psi)).$$

Finally applying schema (**K**) to the right hand side, together with the transitivity of implication and (**MP**) we get

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa * \overline{c}, \phi \to \psi) \to \mathtt{ist}(\kappa * \overline{c}, \phi) \to \mathtt{ist}(\kappa * \overline{c}, \psi)$$

thus proving the inductive step. $\Box_{\mathrm{K}*}$

Proof $(\triangle^*)$: The proof is an induction on the length of the context sequence $\overline{c}$. The base case is simply $(\triangle)$. The inductive step is proved by starting from the inductive hypothesis

$$\vdash_{\overline{\kappa}*\kappa} \mathtt{ist}(\overline{c}, \mathtt{ist}(\kappa_1, \phi) \vee \psi) \to \mathtt{ist}(\overline{c}, \mathtt{ist}(\kappa_1, \phi)) \vee \mathtt{ist}(\overline{c}, \psi)$$

and applying the context rule (**CS**) and the axiom schema (**K**) we get

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa * \overline{c}, \mathtt{ist}(\kappa_1, \phi) \vee \psi) \to \mathtt{ist}(\kappa, \mathtt{ist}(\overline{c}, \mathtt{ist}(\kappa_1, \phi)) \vee \mathtt{ist}(\overline{c}, \psi)).$$

The right hand side can be rewritten using ($\triangle$), and applying the transitivity of implication and (**MP**) results in

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa * \overline{c}, \mathtt{ist}(\kappa_1, \phi) \vee \psi) \rightarrow \mathtt{ist}(\kappa * \overline{c}, \mathtt{ist}(\kappa_1, \phi)) \vee \mathtt{ist}(\kappa * \overline{c}, \psi)$$

which proves the inductive step. $\square_{\triangle^*}$

The proof of ($\triangle^*_-$) is analogous to the above proof of ($\triangle^*$). The proof of (CS*) is trivial by applying the context switching rule (**CS**) once for every context in the context sequence $\overline{c}$.

2.4.1 Conjunctive Normal Forms

In this section we show that any formula is provably equivalent to one in a certain syntactic form. This equivalence plays an important role in the completeness proof.

Definition (CNF): A formula ϕ is in conjunctive normal form (CNF) iff it is of the form $E_1 \wedge E_2 \wedge \cdots \wedge E_l$, and each E_i is of the form $\alpha_{i1} \vee \alpha_{i2} \vee \cdots \vee \alpha_{ir_i}$, where each α_{ij} is either a literal (cf. §2.1) or $\mathtt{ist}^{\pm}(\overline{c}, \beta)$ for some disjunction of literals β. Note that r_i and l can be 1.

Lemma (CNF): For any formula ϕ and context sequence $\overline{\kappa}$ which satisfy the definedness conditions ($\mathrm{Vocab}(\overline{\kappa}, \phi) \subseteq \mathsf{Vocab}$), there exists a formula ϕ^* which is in CNF, such that $\vdash_{\overline{\kappa}} \phi \leftrightarrow \phi^*$.

Proof (CNF): The proof is by induction on n, the maximum nesting of $\mathtt{ist}$'s in the formula ϕ. The base case where $n = 0$ is trivial since $\phi \leftrightarrow \phi^*$ is an instance of a tautology, and $\mathrm{Vocab}(\overline{\kappa}, \phi) = \mathrm{Vocab}(\overline{\kappa}, \phi^*)$, and therefore $\vdash_{\overline{\kappa}} \phi \leftrightarrow \phi^*$ is an axiom. Assume the lemma is true for formulas whose maximal nesting of $\mathtt{ist}$'s is n. We consider two cases. Let ϕ be a formula with $n + 1$ nestings of $\mathtt{ist}$'s. First assume that ϕ is of the form $\mathtt{ist}(\kappa, \psi)$. Since ψ has maximum n nested $\mathtt{ist}$'s, by inductive hypothesis we can assume that $\psi = \psi^*$ is in CNF. The $\mathtt{ist}$ can be propagated through the conjunctions in ψ^* using (**M**) and (**C**) axiom schemas, and through the relevant disjunctions by using the ($\triangle$) axiom schema. More formally, assume

$$\psi^* = \bigwedge_i \bigvee_j \alpha_{ij}$$

where each α_{ij} is a literal, $\mathtt{ist}(\overline{c}, \beta)$, or $\neg\mathtt{ist}(\overline{c}, \beta)$, for some disjunction of literals β. Then $\mathtt{ist}(\kappa, \psi^*)$ is

$$\mathtt{ist}(\kappa, \bigwedge_i \bigvee_j \alpha_{ij}).$$

By the (**M**) axiom schema this is equivalent to

$$\bigwedge_i \mathtt{ist}(\kappa, \bigvee_j \alpha_{ij}).$$

This can be rewritten using the commutative and associative properties of disjunction to

$$\bigwedge_i \mathtt{ist}(\kappa, \mathtt{ist}^{\pm}(\overline{c}_{i1}, \beta_{i1}) \vee \cdots \vee \mathtt{ist}^{\pm}(\overline{c}_{il}, \beta_{il}) \vee \beta_{il+1})$$

where β_{ij} is a disjunction of literals. If $l = 0$, then the above formula is in CNF. If $l \geq 1$, then the $(\triangle)$, $(\triangle_{-})$, and **(Or)** axiom schemas allow us to propagate the `ist` through the disjunction resulting in

$$\bigwedge_i \mathtt{ist}(\kappa, \mathtt{ist}^{\pm}(\overline{c}_{i1}, \beta_{i1})) \vee \cdots \vee \mathtt{ist}(\kappa, \mathtt{ist}^{\pm}(\overline{c}_{il}, \beta_{il})) \vee \mathtt{ist}(\kappa, \beta_{il+1}).$$

The latter is of course

$$\bigwedge_i \mathtt{ist}^{\pm}(\kappa * \overline{c}_{i1}, \beta_{i1}) \vee \cdots \vee \mathtt{ist}^{\pm}(\kappa * \overline{c}_{il}, \beta_{il}) \vee \mathtt{ist}(\kappa, \beta_{il+1}).$$

In the general case we can write ϕ as $\mu(\theta_1, \ldots, \theta_n)$, where $\theta_i = \mathtt{ist}^{\pm}(\overline{c}_i, \chi_i)$ and $\phi = \mu(p_1, \ldots, p_n)$ is an element of PROP, for some new distinct propositional atoms $p_1, \ldots, p_n$. By the above and **(REP)** we have $\vdash_{\overline{\kappa}} \mu(\theta_1, \ldots, \theta_n) \leftrightarrow \mu(\theta_1^*, \ldots, \theta_n^*)$, where θ_i^* is the CNF of θ_i. We may write $\mu(\theta_1^*, \ldots, \theta_n^*)$ as $\chi(\xi_1, \ldots, \xi_m)$ where $\xi_i = \mathtt{ist}^{\pm}(\overline{c}_i, \pi_i)$ and $\chi(q_1, \ldots, q_m) \in$ PROP for some new propositional atoms $q_1, \ldots, q_m$. We now let $\chi^*(q_1, \ldots, q_m)$ be the CNF of $\chi(q_1, \ldots, q_m)$. By **(PL)** we have $\vdash_{\overline{\kappa}} \chi(\xi_1, \ldots, \xi_m) \leftrightarrow \chi^*(\xi_1, \ldots, \xi_m)$, and furthermore $\vdash_{\overline{\kappa}} \phi \leftrightarrow \chi^*(\xi_1, \ldots, \xi_m)$. Since π_i is a disjunction of literals and thus $\chi^*(\xi_1, \ldots, \xi_m)$ is in CNF, we conclude that $\chi^*(\xi_1, \ldots, \xi_m)$ is ϕ^* $\square_{\mathbf{CNF}}$

2.5 Soundness

In this section we demonstrate the soundness of the system.

Theorem (soundness): If $\vdash_{\overline{\kappa}} \phi$, then for any model $\mathfrak{M}$ which is classical on Vocab we have that $\mathfrak{M} \models_{\overline{\kappa}} \phi$. Furthermore, if T $\vdash_{\overline{\kappa}} \phi$, then for any model $\mathfrak{M}$ which is classical on Vocab we have that if for all $\psi \in$ T $\mathfrak{M} \models_{\overline{\kappa}} \psi$, then $\mathfrak{M} \models_{\overline{\kappa}} \phi$.

Proof (soundness): We need to show that instances of all the axiom schemas are valid and that the inference rules preserve validity.

Case PROP: $\mathfrak{M} \models_{\overline{\kappa}} \phi$, provided ϕ is an instance of a tautology and Vocab$(\overline{\kappa}, \phi) \subseteq$ *Vocab*$(\mathfrak{M})$. When $\mathfrak{M}$ is classical on ϕ the satisfaction relation for implication and negation is defined same as for classical propositional logic.

Case K: $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi \to \psi) \to \mathtt{ist}(\kappa_1, \phi) \to \mathtt{ist}(\kappa_1, \psi)$, provided Vocab$(\overline{\kappa} * \kappa_1, \phi \to \psi) \subseteq$ *Vocab*$(\mathfrak{M})$. Assume $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi \to \psi)$ and $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi)$, for $\mathfrak{M}$ classical on Vocab$(\overline{\kappa} * \kappa_1, \phi \to \psi)$. Then from the definition of satisfaction for `ist` it follows that $(\forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}})(\mathfrak{M}, \nu_1 \models_{\overline{\kappa} * \kappa_1} \phi \to \psi)$ and $(\forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}})(\mathfrak{M}, \nu_1 \models_{\overline{\kappa} * \kappa_1} \phi)$. By the definition of satisfaction for implication it follows that $(\forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}})$ $(\mathfrak{M}, \nu_1 \models_{\overline{\kappa} * \kappa_1} \psi)$, and again by the definition of satisfaction for the `ist` modality that $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \psi)$.

Case $\triangle$: $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \vee \psi) \to \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \vee \mathtt{ist}(\kappa_1, \psi)$, provided

$$\text{Vocab}(\overline{\kappa}, \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \vee \psi)) \subseteq \textsf{Vocab}.$$

For $\mathfrak{M}$ classical on

$$\text{Vocab}(\overline{\kappa}, \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \vee \psi))$$

assume that

$$\mathfrak{M} \models_{\overline{\kappa}} \ \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi) \vee \psi).$$

Now by the definition of satisfaction for $\mathtt{ist}$ it follows that

$$(\forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}}) \ (\mathfrak{M}, \nu_1 \models_{\overline{\kappa}*\kappa_1} \ \mathtt{ist}(\kappa_2, \phi) \vee \psi).$$

Again by the definition of satisfaction for $\mathtt{ist}$ it follows that

$$(\forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}})(((\forall \nu_2 \in (\overline{\kappa} * \kappa_1 * \kappa_2)^{\mathfrak{M}})(\mathfrak{M}, \nu_2 \models_{\overline{\kappa}*\kappa_1*\kappa_2} \ \phi)) \quad \text{or} \quad \mathfrak{M}, \nu_1 \models_{\overline{\kappa}*\kappa_1} \ \psi).$$

Since ν_1 does not occur in the first disjunct it follows that

$$(\forall \nu_2 \in (\overline{\kappa} * \kappa_1 * \kappa_2)^{\mathfrak{M}}) \ (\mathfrak{M}, \nu_2 \models_{\overline{\kappa}*\kappa_1*\kappa_2} \ \phi) \quad \text{or} \quad (\forall \nu_1 \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}}) \ (\mathfrak{M}, \nu_1 \models_{\overline{\kappa}*\kappa_1} \ \psi).$$

So by the definition of satisfaction for $\mathtt{ist}$ this becomes:

$$\mathfrak{M} \models_{\overline{\kappa}} \ \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \vee \mathtt{ist}(\kappa_1, \psi).$$

Next we prove that the inference rules preserve validity.

Case MP: if $\mathfrak{M} \models_{\overline{\kappa}} \phi$ and $\mathfrak{M} \models_{\overline{\kappa}} \phi \rightarrow \psi$ then $\mathfrak{M} \models_{\overline{\kappa}} \psi$. Like in the case of (PROP) we argue that (**MP**) must hold in a context since in a fixed context the satisfaction relation for implication is defined same as for classical propositional logic.

Case CS: if $\mathfrak{M} \models_{\overline{\kappa}*\kappa_1} \phi$ then $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi)$. By definition of validity in a model $\mathfrak{M} \models_{\overline{\kappa}*\kappa_1} \phi$ iff for all $\nu \in (\overline{\kappa} * \kappa_1)^{\mathfrak{M}}$, $\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi$. By definition of satisfaction for $\mathtt{ist}$ this is equivalent to $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi)$ $\Box_{\textbf{soundness}}$

2.6 Completeness

We begin by introducing some concepts needed to state the completeness theorem.

Definition (satisfiability): A set of formulas T is *satisfiable in context* $\overline{\kappa}$ *with vocabulary* Vocab iff there exists a model $\mathfrak{M}$ classical on Vocab, such that for all $\phi \in$ T, $\mathfrak{M} \models_{\overline{\kappa}} \phi$.

Definition (consistency): A formula ϕ is *consistent in* $\overline{\kappa}$ *with* Vocab, where Vocab$(\overline{\kappa}, \phi) \subseteq$ Vocab iff not $\vdash_{\overline{\kappa}} \neg\phi$. A finite set T is consistent in $\overline{\kappa}$ with Vocab iff $\bigwedge\!\!\!\bigwedge$ T, the conjunction of all the formulas in T, is consistent in $\overline{\kappa}$ with Vocab. An infinite set T is consistent in $\overline{\kappa}$ with Vocab iff every finite subset of T is consistent in $\overline{\kappa}$ with Vocab. A set T is inconsistent in $\overline{\kappa}$ with Vocab iff the set T is not consistent in $\overline{\kappa}$ with Vocab.

A set T is maximally consistent in $\overline{\kappa}$ with Vocab iff T is consistent in $\overline{\kappa}$ with Vocab and for all $\phi \notin$ T such that Vocab$(\overline{\kappa}, \phi) \subseteq$ Vocab, T $\cup \{\phi\}$ is inconsistent in $\overline{\kappa}$ with Vocab.

As is usual, an important part of the completeness proof is the Lindenbaum lemma allowing any consistent set of wffs to be extended to a maximally consistent set.

Lemma (Lindenbaum): If T is consistent in $\overline{\kappa}$ with Vocab, then T can be extended to a maximally consistent set T_0 in $\overline{\kappa}$ with Vocab.

Proof (Lindenbaum): Assume T is consistent in $\overline{\kappa}$ with Vocab. Enumerate all the sentences ϕ_i such that $\mathrm{Vocab}(\overline{\kappa}, \phi_i) \subseteq \mathsf{Vocab}$. We define an infinite chain of sets of sentences, $\mathrm{T}_1, \mathrm{T}_2, \ldots$, inductively. $\mathrm{T}_1 := \mathrm{T}$, and T_{i+1} is either $\mathrm{T}_i \cup \{\phi_i\}$ or $\mathrm{T}_i \cup \{\neg\phi_i\}$, whichever is consistent. Note that one has to be consistent, because if both $\mathrm{T}_i \cup \{\phi_i\}$ and $\mathrm{T}_i \cup \{\neg\phi_i\}$ were inconsistent then $\mathrm{T}_i \cup \{\phi_i \vee \neg\phi_i\}$ would also be inconsistent, and since $\phi_i \vee \neg\phi_i$ is a tautology this would mean that T_i was inconsistent. The set $\cup_{i=1}^{\infty}\mathrm{T}_i$ is maximally consistent. $\Box_{\textbf{Lindenbaum}}$

Now we proceed to state and prove the completeness of the system.

Theorem (completeness): For any set of formulas T, if T is consistent in $\overline{\kappa}_0$ with Vocab, then T is satisfiable in $\overline{\kappa}_0$ with Vocab.

Proof (completeness): Assume T is consistent in $\overline{\kappa}_0$ with Vocab. By the (**Lindenbaum lemma**) we can extend T to a maximally consistent set T_0. From T_0 we will construct the model $\mathfrak{M}_0$. For each $\overline{\kappa} = \overline{\kappa}_0 * \overline{c} \in \mathbb{K}^*$ define

$$\mathrm{T}_{\overline{\kappa}+} := \{\phi \,|\, \mathrm{T}_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \phi), \phi \in \mathrm{PROP}\}.$$

Lemma ($\mathrm{T}_{\overline{\kappa}+}$): $\mathrm{T}_{\overline{\kappa}+}$ is closed under logical consequence: for all ϕ where $\mathrm{Vocab}(\overline{\kappa}, \phi) \subseteq \mathsf{Vocab}$, if ϕ tautologically follows from $\mathrm{T}_{\overline{\kappa}+}$ then $\phi \in \mathrm{T}_{\overline{\kappa}+}$.

Note that $\mathrm{T}_{\overline{\kappa}+}$ need not be either maximally consistent or even consistent. Now, using only the sets $\mathrm{T}_{\overline{\kappa}+}$ of formulas from PROP, we will define a model $\mathfrak{M}_0$ for the set of formulas T_0. We define the domain of $\mathfrak{M}_0$

$$\mathrm{Dom}(\mathfrak{M}_0) := \{\overline{\kappa} \,|\; \overline{\kappa} \leq \overline{\kappa}_0, \exists \overline{\kappa}' \in \mathrm{Dom}(\mathsf{Vocab}),\ \overline{\kappa}' \leq \overline{\kappa}\}$$

and for all $\overline{\kappa} \in \mathrm{Dom}(\mathfrak{M}_0)$

$$\mathfrak{M}_0(\overline{\kappa}) := \{\nu \,|\, \mathrm{Dom}(\nu) = \mathsf{Vocab}(\overline{\kappa}),\ \forall \phi \in \mathrm{T}_{\overline{\kappa}+}, \overline{\nu}(\phi) = 1\}.$$

In the above, $\overline{\nu}$ is the unique homomorphic extension of ν with respect to the propositional connectives. All that remains to be shown is that $\mathfrak{M}_0$ as defined is a model, i.e. that it satisfies the two additional conditions imposed in the definition of a model. we first note that it clearly meets condition 1, since all the truth assignments associated with a context must have the same domain. Condition 2 is met since $\mathrm{Dom}(\mathfrak{M}_0)$ as defined is a subtree rooted at $\overline{\kappa}_0$. Note that if $\mathrm{T}_{\overline{\kappa}+}$ is empty (which corresponds to the case where $\mathsf{Vocab}(\overline{\kappa}) = \emptyset$), then $\mathfrak{M}_0(\overline{\kappa})$ is a singleton set, whose only member is the *empty truth assignment*. Finally, to establish completeness we need only prove the truth lemma. The proof of the truth lemma is based on the CNF construction and is the novel aspect of this completeness proof.

Lemma (truth): For any ϕ such that $\mathrm{Vocab}(\overline{\kappa}_0, \phi) \subseteq \mathsf{Vocab}$,

$$\phi \in \mathrm{T}_0 \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi.$$

Clearly, if $\phi \in \mathrm{T}$, then also $\phi \in \mathrm{T}_0$, and therefore by truth lemma we get $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi$. Note that in the case where T is a single formula, ϕ, which satisfies the definedness conditions, the contrapositive of the conclusion is: $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \neg\phi$ implies $\vdash_{\overline{\kappa}} \neg\phi$. $\square_{\mathbf{completeness}}$

2.6.1 Proof of the Truth Lemma

Before we give the proof of the truth lemma, we need to state a property of the model $\mathfrak{M}_0$ that is needed in the `ist` case of the truth lemma.

Lemma ($\mathfrak{M}_0$): Let $\mathfrak{M}_0$ be a model as defined from T_0 in the completeness proof. Then for all $\phi \in \mathrm{PROP}$ where $\mathrm{Vocab}(\overline{\kappa}_0 * \overline{c}, \phi) \subseteq \mathsf{Vocab}$,

$$\mathrm{T}_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \phi) \text{ iff for all } \nu \in \mathfrak{M}_0(\overline{\kappa}_0 * \overline{c}) \quad \nu(\phi) = 1.$$

A frequently used instance of the $\mathfrak{M}_0$ lemma is that $\mathrm{T}_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \phi \wedge \neg\phi)$ iff $\mathfrak{M}_0(\overline{\kappa}_0 * \overline{c}) = \emptyset$, for all ϕ satisfying the (**definedness condition**).

Proof (truth lemma): Instead of proving $\phi \in \mathrm{T}_0$ iff $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi$ we will prove the statement

(TL) $\quad (\forall \psi \in \mathbb{W})\psi$ is in CNF $\quad$ implies $\quad (\psi \in \mathrm{T}_0 \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \psi)$.

To see that the former follows from the latter, assume $\phi \in \mathrm{T}_0$. By the (**CNF lemma**), there exists formula ϕ^* in CNF such that $\vdash_{\overline{\kappa}_0} \phi \leftrightarrow \phi^*$. Using maximal consistency of T_0, it follows that $\phi^* \in \mathrm{T}_0$. Therefore, by (**TL**) it must be the case that $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi^*$. Our logic is sound: $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi^*$ iff $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi$, and thus we conclude that $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \phi$. We can simply reverse the steps of the argument to prove the other direction of the biconditional.

We prove (**TL**) by induction on the structure of the formula ψ. In the base case ψ is an atom, and thus in CNF. From the definition of $\mathfrak{M}_0(\overline{\kappa}_0)$ it follows that $\rho \in \mathrm{T}_0 \Leftrightarrow \mathfrak{M}_0 \models_{\overline{\kappa}_0} \rho$. In proving the inductive step we first examine $\psi = \chi \vee \mu$. The inductive hypothesis is that the lemma is true for formulas χ and μ. Assume $\chi \vee \mu$ is in CNF. Then both χ and μ must also be in CNF. Since T_0 is maximally consistent $\chi \vee \mu \in \mathrm{T}_0$ iff either $\chi \in \mathrm{T}_0$ or $\mu \in \mathrm{T}_0$. By the inductive hypothesis this will be true iff either $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \chi$ or $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mu$, and by the definition of satisfaction iff $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \chi \vee \mu$. The inductive step for conjunction and negation is similar. We make use of the fact that if $\chi \wedge \mu$ is in CNF, then so are both χ and μ; and if $\neg\chi$ is in CNF, then so is χ. The interesting case is when ψ is an `ist` formula. Assume that ψ is in CNF. Then ψ must be of the form

$$\psi = \mathtt{ist}^{\pm}(\overline{c}, \chi),$$

where χ is a disjunction of literals. The context sequence $\overline{c}$ will sometimes be written as $\kappa_1 * \cdots * \kappa_n$. We will examine two cases, depending on whether or not any of the sets of sentences $\mathrm{T}_{(\overline{\kappa}_0 * \overline{c}')+}$ where $\overline{c} \leq \overline{c}'$, are inconsistent. The sets $\mathrm{T}_{(\overline{\kappa}_0 * \overline{c}')+}$, where $\overline{c} \leq \overline{c}'$, are all consistent iff the formula

($\mathrm{D}_{\overline{c}}$) $\quad \mathtt{ist}(\overline{c}, \neg\phi) \rightarrow \neg\mathtt{ist}(\overline{c}, \phi)$

is in T_0, for any wff ϕ which satisfies the definedness condition. The proof of this is identical to the soundness and completeness proofs of a context system with axiom schema (**D**) w.r.t. the set of consistent models, given in §3.3. Formula ($D_{\overline{c}}$) is equivalent to

$$\neg\mathtt{ist}(\overline{c}, \phi \wedge \neg\phi) \in T_0,$$

for all ϕ satisfying the definedness condition; the proof carries over from normal systems of modal logic. Now we state a useful consequence of ($D_{\overline{c}}$)'s.

Lemma ($D_{\overline{c}}$):

Let $\overline{c}$ be $\kappa_1 * \cdots * \kappa_n$. If $D_{(\kappa_1 * \cdots * \kappa_{n-1})} \in T_0$, then

$$\mathtt{ist}^{\pm}(\overline{c}, \phi) \in T_0 \quad \text{iff} \quad \pm\,\mathtt{ist}(\overline{c}, \phi) \in T_0$$

for any formula ϕ which satisfies the definedness convention. The sign on the right hand side is positive iff there is an even number of negations in the $\mathtt{ist}^{\pm}$ on the left hand side.

Now we examine the two cases need to prove the inductive step for `ist` case of the truth lemma.

Case $D_{(\kappa_1 * \cdots * \kappa_{n-1})} \in T_0$: In this case we assume $D_{(\kappa_1 * \cdots * \kappa_{n-1})} \in T_0$ and that $\psi \in T_0$. Then by the $D_{\overline{c}}$ lemma:

$$\mathtt{ist}^{\pm}(\overline{c}, \chi) \in T_0 \quad \text{iff} \quad \pm\,\mathtt{ist}(\overline{c}, \chi) \in T_0.$$

We first consider the positive case.

$$\mathtt{ist}(\overline{c}, \chi) \in T_0 \quad \text{iff} \quad T_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi).$$

Now by ($\mathfrak{M}_0$ **lemma**) and the definedness condition $\mathrm{Vocab}(\overline{\kappa}_0 * \overline{c}) \subseteq \mathsf{Vocab}$ we have

$$T_0 \vdash_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi) \quad \text{iff} \quad (\forall \nu \in \mathfrak{M}_0(\overline{\kappa}))(\overline{\nu}(\chi) = 1).$$

By the definition of satisfaction:

$$(\forall \nu \in \mathfrak{M}_0(\overline{\kappa}))(\overline{\nu}(\chi) = 1) \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi).$$

Now since $D_{(\kappa_1 * \cdots * \kappa_{n-1})} \in T_0$, and by ($\mathfrak{M}_0$ **lemma**) we obtain:

$$\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi) \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}^{\pm}(\overline{c}, \chi).$$

The negative case where $\neg\mathtt{ist}(\overline{c}, \chi) \in T_0$ follows since

$$\neg\mathtt{ist}(\overline{c}, \chi) \in T_0 \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \neg\mathtt{ist}(\overline{c}, \chi),$$

is equivalent to

$$\mathtt{ist}(\overline{c}, \chi) \in T_0 \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\overline{c}, \chi),$$

by the maximal consistency of T_0, the definition of the satisfaction relation, and the usual definedness conditions.

Case $D_{(\kappa_1 * \cdots * \kappa_{n-1})} \notin T_0$: In the second case, $D_{(\kappa_1 * \cdots * \kappa_{n-1})} \notin T_0$. Let j be the index of the first inconsistent context; formally $D_{(\kappa_1 * \cdots * \kappa_j)} \notin T_0$ and $D_{(\kappa_1 * \cdots * \kappa_{j-1})} \in T_0$. Then for all ϕ satisfying the definedness condition we have $\neg\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi \wedge \neg\phi) \notin \mathrm{T}_0$. Now by maximal consistency of T_0:

$$\neg\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi \wedge \neg\phi) \notin \mathrm{T}_0 \,\text{iff}\, \mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi \wedge \neg\phi) \in \mathrm{T}_0.$$

Furthermore, for all ψ satisfying the definedness condition, by (K*) and (**MP**)

$$\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi \wedge \neg\phi) \in \mathrm{T}_0 \quad \text{iff} \quad \mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \psi) \in \mathrm{T}_0.$$

Thus, $\mathrm{T}_{(\overline{\kappa}_0 * \kappa_1 * \cdots * \kappa_j)+}$ is inconsistent, $\mathfrak{M}_0(\overline{\kappa}_0 * \kappa_1 * \cdots * \kappa_j) = \emptyset$, and consequently

$$\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi) \in \mathrm{T}_0 \quad \text{iff} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \phi)$$

for all ϕ such that $\mathrm{Vocab}(\overline{\kappa}_0 * \kappa_1 * \cdots * \kappa_j, \phi) \subseteq \mathsf{Vocab}$.

Now assume that $\psi \in \mathrm{T}_0$. Since $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{j-1})} \in \mathrm{T}_0$, we may conclude using the ($\mathrm{D}_{\overline{c}}$ **lemma**) that:

$$\begin{aligned}&\mathtt{ist}^{\pm}(\overline{c}, \chi) \in \mathrm{T}_0 \quad \text{iff} \\ &\pm\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \pm\mathtt{ist}^{\pm}(\kappa_{j+1} * \cdots * \kappa_n, \chi)) \in \mathrm{T}_0.\end{aligned}$$

Now, we consider the positive case:

$$\mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \pm\mathtt{ist}^{\pm}(\kappa_{j+1} * \cdots * \kappa_n, \chi)) \in \mathrm{T}_0$$

which is equivalent to, due to the above property of the context sequence $\kappa_1 * \cdots * \kappa_j$,

$$\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\kappa_1 * \cdots * \kappa_j, \pm\mathtt{ist}^{\pm}(\kappa_{j+1} * \cdots * \kappa_n, \chi)).$$

Since $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{j-1})} \in \mathrm{T}_0$ and by ($\mathfrak{M}_0$ **lemma**), this is equivalent to:

$$\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}^{\pm}(\overline{c}, \chi).$$

The negative case is reduced to the positive case analogously to the negative case of the $\mathrm{D}_{(\kappa_1 * \cdots * \kappa_{n-1})} \in \mathrm{T}_0$ case. Note that in the entire proof of the inductive step for `ist` we did not need the inductive hypothesis, making use only of the special form of χ which is guaranteed because ψ is in CNF. $\Box_{\textbf{truth-lemma}}$

2.6.2 Proof of the Minor Lemmas

Proof ($\mathrm{D}_{\overline{c}}$ lemma): First note that the formula $\mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \cdots \mathtt{ist}(\kappa_n, \phi) \cdots))$, which has only positive occurrences of `ist`'s, can be written as $\mathtt{ist}(\overline{c}, \phi)$. Therefore the formula $\mathtt{ist}^{\pm}(\overline{c}, \phi)$ can be written as

$$\mathtt{ist}(\overline{c}_1, \neg\mathtt{ist}(\overline{c}_2, \neg\mathtt{ist}(\overline{c}_3, \cdots \neg\mathtt{ist}(\overline{c}_j, \phi) \cdots))),$$

where $\overline{c} = \overline{c}_1 * \cdots * \overline{c}_j$. Now assume that $\mathtt{ist}^{\pm}(\overline{c}, \phi) \in \mathrm{T}_0$. Then using this fact we have

$$\mathtt{ist}(\overline{c}_1, \neg\mathtt{ist}(\overline{c}_2, \cdots \neg\mathtt{ist}(\overline{c}_{j-1}, \neg\mathtt{ist}(\overline{c}_j, \phi) \cdots))) \in \mathrm{T}_0.$$

This is in turn equivalent to

$$\neg\mathtt{ist}(\overline{c}_1, \mathtt{ist}(\overline{c}_2, \cdots \neg\mathtt{ist}(\overline{c}_{j-1}, \neg\mathtt{ist}(\overline{c}_j, \phi) \cdots))) \in \mathrm{T}_0$$

since $D_{\overline{c}_1} \in T_0$, and applying (K*) and (**MP**). Repeating this procedure (using the fact that $D_{\overline{c}_2} \in T_0, \ldots, D_{\overline{c}_{j-1}} \in T_0$) we can "bubble" out the relevant negations to obtain one direction of the desired result:

$$\text{if} \quad \texttt{ist}^{\pm}(\overline{c}, \phi) \in T_0 \quad \text{then} \quad \begin{cases} \texttt{ist}(\overline{c}_1 * \cdots * \overline{c}_j, \phi) \in T_0, & \text{if } j \text{ is even} \\ \neg\texttt{ist}(\overline{c}_1 * \cdots * \overline{c}_j, \phi) \in T_0, & \text{if } j \text{ is odd} \end{cases}$$

The other direction follows from (**ND**). $\square_{D_{\overline{c}}}$

Proof ($T_{\overline{\kappa}+}$ lemma): To prove that $T_{\overline{\kappa}+}$ is closed under logical consequence, assume ϕ tautologically follows from $T_{\overline{\kappa}+}$. Then by the compactness of propositional logic there exist formulas $\phi_0, \ldots, \phi_n$ in $T_{\overline{\kappa}+}$ such that $\{\phi_0, \ldots, \phi_n\}$ tautologically imply ϕ. By the axiom schema (**C**), $T_{\overline{\kappa}+}$ is closed under conjunction. Therefore, we get $\psi = \phi_0 \wedge \ldots \wedge \phi_n \in T_{\overline{\kappa}+}$, and by the definition of $T_{\overline{\kappa}+}$, this means that $\texttt{ist}(\overline{c}, \psi) \in T_0$, where $\overline{\kappa} = \overline{\kappa}_0 * \overline{c}$. Therefore $\psi \rightarrow \phi$ is a tautology, and since $\text{Vocab}(\overline{\kappa}, \psi \rightarrow \phi) \subseteq \mathsf{Vocab}$, it is also an axiom in $\overline{\kappa}$; formally $\vdash_{\overline{\kappa}} \psi \rightarrow \phi$. Applying the inference rule (CS*) we get $\vdash_{\overline{\kappa}_0} \texttt{ist}(\overline{c}, \psi \rightarrow \phi)$, and together with $\texttt{ist}(\overline{c}, \psi) \in T_0$ and axiom schema (**K**) it must be the case that $\texttt{ist}(\overline{c}, \phi) \in T_0$. $\square_{T_{\overline{\kappa}+}}$

Proof ($\mathfrak{M}_0$ lemma):

Case $\Rightarrow$: Trivial from the construction of $\mathfrak{M}_0$.

Case $\Leftarrow$: Suppose $\phi \in \text{PROP}$, $\text{Vocab}(\overline{\kappa}, \phi) \subseteq \mathsf{Vocab}$, and $T_0 \not\vdash_{\overline{\kappa}_0}^{\mathsf{Vocab}} \texttt{ist}(\overline{c}, \phi)$. Then by maximal consistency of T_0, $\neg\texttt{ist}(\overline{c}, \phi) \in T_0$. By $T_{\overline{\kappa}+}$ lemma, we get that ϕ does not tautologically follow from $T_{\overline{\kappa}+}$ or equivalently $T_{\overline{\kappa}+} \cup \{\neg\phi\}$ is consistent. Therefore, there is a truth assignment which satisfies all the formulas in $T_{\overline{\kappa}+}$ but does not satisfy ϕ. Thus there exists a $\nu \in \mathfrak{M}_0(\overline{\kappa})$ with $\nu(\phi) = 0$. $\square_{\mathfrak{M}_0}$

3 Correspondence Results

In this section we provide soundness and completeness results for several extensions of the system described in §2. The extensions correspond to certain intuitive principles concerning the nature of contexts. In each extension the syntax and semantics are the same as in the general case described in §2, and the (**definedness convention**) still holds. Only the class of models and axioms are modified.

3.1 Consistency

Sometimes it is desirable to ensure that all contexts are consistent.

In this system we examine the class of *consistent models*, $\mathfrak{Consistent}$. A model $\mathfrak{M} \in \mathfrak{Consistent}$ iff for any context sequence $\overline{\kappa} \in \text{Dom}(\mathfrak{M})$, such that $\overline{\kappa} < \overline{\kappa}_0$,

$$\mathfrak{M}(\overline{\kappa}) \neq \emptyset,$$

where $\overline{\kappa}_0$ is the root of the subtree of the domain of $\mathfrak{M}$. The additional restriction that $\overline{\kappa} < \overline{\kappa}_0$ is needed because our language does not allow us to express facts about the consistency of the root context $\overline{\kappa}_0$. The following axiom schema is sound with respect to the class of consistent models $\mathfrak{Consistent}$:

(D) $\vdash_{\overline{\kappa}}$ $\mathtt{ist}(\kappa, \neg\phi) \rightarrow \neg\mathtt{ist}(\kappa, \phi)$

for any $\overline{\kappa} \in \mathrm{Dom}(\mathfrak{M})$, provided the usual definedness convention is satisfied. Axiom schema (D) is also commonly used in modal logic, and it is sound and complete for the set of serial Kripke frames, in which for each world there is another world from which it is accessible from. Note that axiom (D) is equivalent to

$$\vdash_{\overline{\kappa}} \neg\mathtt{ist}(\kappa, \phi \wedge \neg\phi).$$

Theorem (completeness): The general context system with the (D) axiom schema is complete with respect to the set of models $\mathfrak{Consistent}$.

Proof (completeness): The proof is based on the completeness proof for the general system. To prove completeness we need to show that given a consistent set of sentences T_0 in context $\overline{\kappa}_0$, the model $\mathfrak{M}_0$ as defined in the completeness proof is a consistent model; formally $\mathfrak{M}_0 \in \mathfrak{Consistent}$. Assume that for some context sequence $\overline{\kappa} * \kappa$ $\mathfrak{M}(\overline{\kappa} * \kappa) = \emptyset$. Then by the definition of satisfaction it follows that $\mathfrak{M}_0 \models_{\overline{\kappa} * \kappa} \phi \wedge \neg\phi$ for any formula ϕ which satisfies the definedness condition. Note that due to the definition of the domain of the model $\mathfrak{M}_0$, for any context in the domain of $\mathfrak{M}_0$ there will always exist a formula ϕ which satisfies the definedness condition. By the context switching rule it follows that $\mathfrak{M}_0 \models_{\overline{\kappa}} \mathtt{ist}(\kappa, \phi \wedge \neg\phi)$, and thus by axiom schema (**M**) it must be the case that $\mathfrak{M}_0 \models_{\overline{\kappa}} \mathtt{ist}(\kappa, \phi) \wedge \mathtt{ist}(\kappa, \neg\phi)$, which contradicts the axiom schema (D). $\Box_{\mathbf{completeness}}$

3.2 Truth

It might be more intuitive to define the `ist` modality to correspond to truth rather than validity; incidently this is also where the `ist` predicate got its name: `is true`. Truth based interpretation of the basic context modality also corresponds to the original suggestions by McCarthy [18]. In this case a context is associated with a single truth assignment rather than a set of truth assignments.

We examine the class of *truth models*, $\mathfrak{Truth}$. A model $\mathfrak{M}$ is a truth model, formally $\mathfrak{M} \in \mathfrak{Truth}$, iff for any context sequence $\overline{\kappa} \in \mathrm{Dom}(\mathfrak{M})$, such that $\overline{\kappa} < \overline{\kappa}_0$,

$$|\, \mathfrak{M}(\overline{\kappa}) \,| \leq 1.$$

The following axiom schema is sound with respect to the class of truth models $\mathfrak{Truth}$:

(Tr) $\vdash_{\overline{\kappa}}$ $\mathtt{ist}(\kappa, \phi) \vee \mathtt{ist}(\kappa, \neg\phi)$

$\overline{\kappa} \in \mathrm{Dom}(\mathfrak{M})$, provided the usual definedness convention is satisfied. Note that (**Tr**) is the converse of (**D**). Also note that a weak form of truth, namely (**ND**), is equivalent to ($\triangle$).

Theorem (completeness): The general context system with the (**Tr**) axiom schema is complete with respect to the set of truth models $\mathfrak{Truth}$.

Proof (completeness): The proof is based on the completeness proof for the general system. To prove the completeness we need to show that given a consistent set of sentences T_0 in context $\overline{\kappa}_0$, the model $\mathfrak{M}_0$ as defined in the completeness proof is a truth model. Assume that $\mathfrak{M}_0$ is not a truth model. Then for some context sequence $\overline{\kappa} \in \mathrm{Dom}(\mathfrak{M}_0)$ there are at least two truth assignments,

$$\{\nu_1, \nu_2\} \subseteq \mathfrak{M}(\overline{\kappa}), \quad \nu_1 \neq \nu_2.$$

Therefore, for some propositional atom ρ it must be the case that $\nu_1(\rho) \neq \nu_2(\rho)$, and by the definition of satisfaction it follows that $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \neg\mathtt{ist}(\kappa, \rho) \wedge \neg\mathtt{ist}(\kappa, \neg\rho)$, which contradicts the soundness of axiom (Tr). $\square_{\mathbf{completeness}}$

Previously we said that $(\triangle)$ is derivable in a system which contains (**D**) and (**Tr**). In fact, a stronger formula is true of this system:

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa, \phi \vee \psi) \leftrightarrow (\mathtt{ist}(\kappa, \phi) \vee \mathtt{ist}(\kappa, \psi)).$$

3.3 Flatness

For some applications all contexts will be identical regardless of where they are examined from. This type of situation will often arise when we use a number of independent databases. For example, if I am booked on flight 921 in the context of the Northwest airlines database, then regardless of which travel agent I choose, in the context of that travel agent, it is true that in the context of Northwest airlines I am booked on flight 921.

In this system we examine a class of what we call *flat models*, $\mathfrak{Flat}$. A model $\mathfrak{M}$ is flat, formally $\mathfrak{M} \in \mathfrak{Flat}$, iff $\mathrm{Dom}(\mathfrak{M}) = \mathbb{K}^*$ and for any context sequences $\overline{\kappa}_1$ and $\overline{\kappa}_2$, and any context κ,

$$\mathfrak{M}(\overline{\kappa}_1 * \kappa) = \mathfrak{M}(\overline{\kappa}_2 * \kappa).$$

When dealing with flat models it might be more intuitive to think of individual contexts rather then context sequences. In that case $\mathfrak{M} \in \mathfrak{Flat}$ can be viewed as a function which maps contexts to finite sets of partial truth assignments, in other words

$$\mathfrak{M} \in \mathbb{K} \cup \{\epsilon\} \mapsto \mathbf{P}(\mathbb{P} \rightarrow_{\mathrm{p}} 2)$$

with the side condition of general models that still applies:

$$(\forall \overline{\kappa} \in \mathbb{K} \cup \{\epsilon\})(\forall \nu_1, \nu_2 \in \mathfrak{M}(\overline{\kappa}))(\mathrm{Dom}(\nu_1) = \mathrm{Dom}(\nu_2))$$

The following flatness axiom schemas are sound with respect to the class of flat models $\mathfrak{Flat}$:

$(4^{-1})\quad \vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_2, \mathtt{ist}(\kappa_1, \phi)) \rightarrow \mathtt{ist}(\kappa_1, \phi)$

$(5^{-1})\quad \vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_2, \neg\mathtt{ist}(\kappa_1, \phi)) \rightarrow \neg\mathtt{ist}(\kappa_1, \phi)$

providing the vocabulary also satisfies the flatness condition: for any context sequences $\overline{\kappa}_1$ and $\overline{\kappa}_2$, and any context κ,

$$\mathsf{Vocab}(\overline{\kappa}_1 * \kappa) = \mathsf{Vocab}(\overline{\kappa}_2 * \kappa).$$

The converse of the flatness axiom schema (4^{-1}), schema (4), corresponds of the modal logic axiom schema S4 (provided that κ_1 is the same as κ_2). Similarly, the converse of (5^{-1}), schema (5), corresponds to the modal logic axiom schema S5. Note that both the schema (4) and the schema (5) are theorems in our system. We first show the derivation of schema (5). Start with the tautology

$$\mathtt{ist}(\kappa, \phi) \vee \neg\mathtt{ist}(\kappa, \phi)$$

and derive

$$\mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa, \phi)) \vee \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa, \phi))$$

by (**CS**), ($\triangle$), and (**MP**). Then, by (4^{-1}), (**REP**), and (**MP**) it follows that

$$\neg\mathtt{ist}(\kappa, \phi) \rightarrow \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa, \phi))$$

which is schema (5). Now we show the derivation of schema (4). Start with the tautology

$$\mathtt{ist}(\kappa, \phi) \vee \neg\mathtt{ist}(\kappa, \phi).$$

Then by (**CS**), ($\triangle$), and (**MP**) we get

$$\mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa, \phi)) \vee \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa, \phi)).$$

This formula, together with the instance of (5^{-1}) yields

$$\mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa, \phi)) \rightarrow \neg\mathtt{ist}(\kappa, \phi)$$

and (**PL**), implies

$$\mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa, \phi)) \vee \neg\mathtt{ist}(\kappa, \phi)$$

which is in fact schema (4).

It is interesting to observe that in every system with (4^{-1}) and (5^{-1}), (**D**) is also derivable. Here is the proof. Start with a tautology

$$\neg(\neg\mathtt{ist}(\kappa, \phi) \wedge \mathtt{ist}(\kappa, \phi)).$$

Therefore

$$\neg(\mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa, \phi)) \wedge \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa, \phi)))$$

by (4^{-1}), (5^{-1}) (4), (5), and (**REP**). Then by (**C**) and (**REP**)

$$\neg\mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa, \phi) \wedge \mathtt{ist}(\kappa, \phi)).$$

Now we apply (**MP**) to the above formula and

$$(\neg \mathtt{ist}(\kappa_1, \neg \mathtt{ist}(\kappa, \phi) \wedge \mathtt{ist}(\kappa, \phi))) \rightarrow (\neg \mathtt{ist}(\kappa_1, \neg \psi \wedge \psi))$$

(which is easily proved providing the vocabulary conditions are met) to derive

$$\neg \mathtt{ist}(\kappa_1, \neg \psi \wedge \psi)$$

which is (**D**). In semantic terms, this means that any flat model is also a consistent model. This is a reasonable property because if a context was inconsistent, then in that context it would be true that all other contexts are also inconsistent. Due to flatness, this would really make all the other contexts inconsistent. Another result which is interesting is that (4^{-1}) is derivable from schema (5) and (**D**); similarly (5^{-1}) is derivable from schema (4) and (**D**).

Theorem (completeness): The general context system with (4^{-1}) and (5^{-1}) axiom schemas is complete with respect to the set of flat models $\mathfrak{Flat}$.

Proof (completeness): The proof is based on the completeness proof for the general system. To prove the completeness we need to show that given a consistent set of sentences T_0 in context $\overline{\kappa}_0$, the model $\mathfrak{M}_0$ as defined in the completeness proof is a flat model; formally $\mathfrak{M}_0 \in \mathfrak{Flat}$. Assume that $\mathfrak{M}_0$ is not flat: for some contexts κ and κ_1,

$$\mathfrak{M}(\overline{\kappa}_0 * \kappa) \neq \mathfrak{M}(\overline{\kappa}_0 * \kappa_1 * \kappa).$$

Therefore there exists some wff ϕ such that

$$(\mathfrak{M}_0 \models_{\overline{\kappa}_0} \neg \mathtt{ist}(\kappa, \phi) \quad \text{and} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0 * \kappa_1} \mathtt{ist}(\kappa, \phi)) \quad \text{or}$$

$$(\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\kappa, \phi) \quad \text{and} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0 * \kappa_1} \neg \mathtt{ist}(\kappa, \phi)).$$

Assume

$$\mathfrak{M}_0 \models_{\overline{\kappa}_0} \neg \mathtt{ist}(\kappa, \phi) \quad \text{and} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0 * \kappa_1} \mathtt{ist}(\kappa, \phi).$$

From the definition of satisfaction for `ist` the second formula is equivalent to

$$\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa, \phi)),$$

and due to the soundness of the (4^{-1}) axiom schema it is equivalent to $\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\kappa, \phi)$, which is a contradiction. A contradiction is derived in a similar fashion using (5^{-1}) when we assume the other disjunct:

$$\mathfrak{M}_0 \models_{\overline{\kappa}_0} \mathtt{ist}(\kappa, \phi) \quad \text{and} \quad \mathfrak{M}_0 \models_{\overline{\kappa}_0 * \kappa_1} \neg \mathtt{ist}(\kappa, \phi).$$

$\square_{\textbf{completeness}}$

4 Meaninglessness as Falsity

In this section we examine a slightly more elaborate modification of the general system introduced in §2. This modification closely models the semantics described but not investigated, in [12]. The general idea here is that if ϕ is not in the vocabulary of κ, then $\mathtt{ist}(\kappa, \phi)$ is taken to be false instead of meaningless or undefined. To cater faithfully to this interpretation, two changes must be made to the semantics of the general system. Firstly, the $\mathtt{ist}$ clause in the definition of Vocab : $\mathbb{K}^* \times \mathbb{W} \to \mathbf{P}(\mathbb{K}^* \times \mathbb{P})$ must be altered to reflect the fact that $\mathtt{ist}(\kappa, \phi)$ will always be in the vocabulary of any context. Secondly, the $\mathtt{ist}$ clause in the definition of satisfaction must also be modified. The appropriate new clause in the definition of Vocab is:

$$\mathrm{Vocab}(\overline{\kappa}, \phi) = \emptyset \text{ if } \phi \text{ is } \mathtt{ist}(\kappa, \phi_0)$$

While the new clause in the definition of satisfaction is:

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \text{ iff } \mathrm{Vocab}(\phi, \overline{\kappa}{*}\kappa_1) \subseteq \mathit{Vocab}(\mathfrak{M}) \text{ and } (\forall \nu_1 \in (\overline{\kappa}{*}\kappa_1)^{\mathfrak{M}})\, \mathfrak{M}, \nu_1 \models_{\overline{\kappa}*\kappa_1} \phi$$

The other clauses in both definitions remain the same, modulo the fact that all occurrences of Vocab in the definition of satisfaction now refer to the new definition. We maintain the (**definedness convention**) in stating the proof system for this version, but again we point out that all occurrences of Vocab now refers to the new definition. The proof system for this version consists of the axioms and rules of the general system, together with the new axiom:

$$(\mathrm{MF}) \qquad \vdash_{\overline{\kappa}} \neg\mathtt{ist}(\kappa_1, \phi) \qquad \text{if } \mathrm{Vocab}(\overline{\kappa} * \kappa_1, \phi) \not\subseteq \mathsf{Vocab}$$

However, there are several important points to observe here. Firstly, because the (**definedness convention**) is substantially more liberal in this version, there will be many more legal instances of tautologies allowed by the schema (**PL**). Indeed in some cases, the (**definedness convention**) will be vacuously true. This is the case with both the ($\triangle$), ($\triangle_{-}$) schemas, and also (modulo the explicit side condition) the (**MF**) schema. Also note that in the rule (**CS**) the (**definedness convention**) holds vacuously for the conclusion of the rule, but not for its hypothesis. Secondly, this new interpretation of Vocab, and its resulting consequences rules out certain previously derivable schemas. For example, (**Or**) schema is no longer derivable in its full generality; the provable version is

$$\vdash_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \phi) \vee \mathtt{ist}(\kappa_1, \psi) \to \mathtt{ist}(\kappa_1, \phi \vee \psi) \qquad \text{if } \mathrm{Vocab}(\overline{\kappa} * \kappa_1, \phi \vee \psi) \subseteq \mathsf{Vocab}.$$

The side-condition is needed to legitimize the appropriate instances of (**PL**) and applications of (**CS**).

The completeness proof for this system is structurally similar to the one given in §2. The only new points are those that arise out of the liberal definition of Vocab.

5 Decidability

The purpose of this section is to show that the propositional logic of contexts is decidable (i.e. that there is an effective procedure that says whether or not a given formula is valid, and hence also a theorem of the system). This will be done by showing that the propositional logic of contexts has the finite model property: any formula that is satisfiable is satisfiable in a model with finitely many finite truth assignments.

Definition (restriction of $\mathfrak{M}(\overline{\kappa})$): We first define the restriction of a single truth assignment, ν, with respect to $\text{Vocab}(\overline{\kappa}_0, \phi)$ to be a truth assignment which, on the atoms that are in the scope of the context sequence $\overline{\kappa}$, corresponds to ν, and is false elsewhere.

$$\nu_{\text{Vocab}(\overline{\kappa}_0,\phi),\overline{\kappa}} = \text{ the unique } \nu' \text{ such that } \nu'(p) = \begin{cases} \nu(p) & <\overline{\kappa}, p> \in \text{Vocab}(\overline{\kappa}_0, \phi) \\ 0 & <\overline{\kappa}, p> \notin \text{Vocab}(\overline{\kappa}_0, \phi) \end{cases}$$

The restriction of a set of truth assignments, V, with respect to $\text{Vocab}(\overline{\kappa}_0, \phi)$ is defined as follows:

$$V_{\text{Vocab}(\overline{\kappa}_0,\phi)} = \{\nu_{\text{Vocab}(\overline{\kappa}_0,\phi),\overline{\kappa}} \,|\, \nu \in V\}.$$

Definition (restriction of $\mathfrak{M}$): The restriction of a model $\mathfrak{M}$ with respect to $\text{Vocab}(\overline{\kappa}_0, \phi)$ is a model which maps every context sequence which appears in ϕ to the set of restricted truth assignments.

$$\mathfrak{M}_{\text{Vocab}(\overline{\kappa}_0,\phi)} : S \to \{\mathfrak{M}(\overline{\kappa})_{\text{Vocab}(\overline{\kappa}_0,\phi)} \,|\, \overline{\kappa} \in \text{Dom}(\text{Vocab}(\overline{\kappa}_0, \phi))\},$$

where S is the set of all subsequences of context sequences from $\text{Dom}(\text{Vocab}(\overline{\kappa}_0, \phi))$.

$$S = \{\overline{\kappa}_1 \,|\, \overline{\kappa}_1 \leq \overline{\kappa}_0 \quad \wedge \quad \exists \overline{\kappa}_2 \in \text{Dom}(\text{Vocab}(\overline{\kappa}_0, \phi)) \quad \overline{\kappa}_2 \leq \overline{\kappa}_1\}.$$

Thus the model $\mathfrak{M}_{\text{Vocab}(\overline{\kappa}_0,\phi)}$ maps a context sequence $\overline{\kappa}$ to a set of truth assignments $\mathfrak{M}(\overline{\kappa})_{\text{Vocab}(\overline{\kappa}_0,\phi)}$:

$$\mathfrak{M}_{\text{Vocab}(\overline{\kappa}_0,\phi)} : \overline{\kappa} \mapsto \mathfrak{M}(\overline{\kappa})_{\text{Vocab}(\overline{\kappa}_0,\phi)}.$$

Theorem (finite model property): $\mathfrak{M} \models_{\overline{\kappa}_0} \phi$ iff $\mathfrak{M}_{\text{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}_0} \phi$.

Note that only the ($\Rightarrow$) direction will be needed to prove the decidability of the propositional logic of context.

Proof (finite model property): We prove a stronger property:

$$\text{if} \quad \text{Vocab}(\overline{\kappa}, \alpha) \subseteq \text{Vocab}(\overline{\kappa}_0, \phi) \quad \text{then} \quad (\mathfrak{M} \models_{\overline{\kappa}} \alpha \quad \text{iff} \quad \mathfrak{M}_{\text{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}} \alpha)$$

by induction on the structure of the formula ϕ.

Case atomic: We need to show that for any atom ρ such that $<\overline{\kappa}, \rho> \in \text{Vocab}(\overline{\kappa}_0, \phi)$ the following holds:

$$\forall \nu \in \mathfrak{M}(\overline{\kappa}) \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \rho \quad \text{iff} \quad \forall \nu \in \mathfrak{M}_{\text{Vocab}(\overline{\kappa}_0,\phi)}(\overline{\kappa}) \quad \mathfrak{M}_{\text{Vocab}(\overline{\kappa}_0,\phi)}, \nu \models_{\overline{\kappa}} \rho.$$

For the ($\Rightarrow$) direction, suppose we are given some truth assignment, $\nu \in \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}(\overline{\kappa})$; we need to show that $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu \models_{\overline{\kappa}} \rho$. By the construction of $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}$ we know that there is a truth assignment $\nu' \in \mathfrak{M}(\kappa)$ such that $\nu'_{\mathrm{Vocab}(\overline{\kappa}_0,\rho),\overline{\kappa}} = \nu$. By assumption we know that $\mathfrak{M}, \nu' \models_{\overline{\kappa}} \rho$, and consequently we know that $<\overline{\kappa}, \rho> \in \mathit{Vocab}(\mathfrak{M})$. Because $<\overline{\kappa}, \rho> \in \mathrm{Vocab}(\overline{\kappa}_0, \phi)$, from the definition of a restriction of a model we conclude that $\rho \in \mathrm{Dom}(\nu)$. Then it must be the case that $\nu(\rho) = \nu'(\rho)$, which implies that $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu \models_{\overline{\kappa}} \rho$.

For the other direction suppose we are given a truth assignment $\nu \in \mathfrak{M}(\overline{\kappa})$; we need to show that $\mathfrak{M}, \nu \models_{\overline{\kappa}} \rho$. We know that there exists a truth assignment $\nu' \in \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}(\overline{\kappa})$ such that $\nu' = \nu_{\mathrm{Vocab}(\overline{\kappa}_0,\rho),\overline{\kappa}}$. From the assumption it follows that $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu' \models_{\overline{\kappa}} \rho$, and therefore we have $\mathfrak{M}, \nu \models_{\overline{\kappa}} \rho$ (since $\rho \in \mathrm{Dom}(\nu)$, which in turn follows from the fact that $\mathrm{Dom}(\nu') \subseteq \mathrm{Dom}(\nu)$).

For the inductive hypothesis in the following three cases assume that the following holds:

$$\text{if} \quad \mathrm{Vocab}(\overline{\kappa}, \alpha) \subseteq \mathrm{Vocab}(\overline{\kappa}_0, \phi) \quad \text{then} \quad (\mathfrak{M} \models_{\overline{\kappa}} \alpha \quad \text{iff} \quad \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}_0} \alpha).$$

Case negation: We need to show that if $\mathrm{Vocab}(\overline{\kappa}, \alpha) \subseteq \mathrm{Vocab}(\overline{\kappa}_0, \phi)$ then

$$(\forall \nu \in \mathfrak{M}(\overline{\kappa}) \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\alpha \quad \text{iff} \quad \forall \nu \in \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}(\overline{\kappa}) \quad \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu \models_{\overline{\kappa}} \neg\alpha).$$

For the ($\Rightarrow$) direction, suppose we are given some truth assignment $\nu \in \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}(\overline{\kappa})$; we need to show that $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu \models_{\overline{\kappa}} \neg\alpha$. By the construction of $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}$ we know that there is a truth assignment $\nu' \in \mathfrak{M}(\overline{\kappa})$ such that $\nu'_{\mathrm{Vocab}(\overline{\kappa}_0,\neg\alpha),\overline{\kappa}} = \nu$. By assumption we know that $\mathfrak{M}, \nu' \models_{\overline{\kappa}} \neg\alpha$, which implies that $\mathfrak{M}, \nu' \not\models_{\overline{\kappa}_0} \alpha$; by the inductive hypothesis we know that $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu \not\models_{\overline{\kappa}} \alpha$. Now because $\mathrm{Vocab}(\overline{\kappa}, \alpha) = \mathrm{Vocab}(\overline{\kappa}, \neg\alpha)$ and because $\mathit{Vocab}(\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)})$ contains all the atoms relevant to ϕ (and hence also to α since $\mathrm{Vocab}(\overline{\kappa}, \alpha) \subseteq \mathrm{Vocab}(\overline{\kappa}_0, \phi)$), we know that the preconditions for the satisfaction relation are met. From this we can conclude that $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu \models_{\overline{\kappa}} \neg\alpha$.

For the other direction suppose we are given a truth assignment $\nu \in \mathfrak{M}(\overline{\kappa})$; we need to show that $\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\alpha$. By the construction of $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}$ we know that there exists a truth assignment $\nu' \in \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}(\overline{\kappa})$ such that $\nu' = \nu_{\mathrm{Vocab}(\overline{\kappa}_0,\phi),\overline{\kappa}}$. By assumption we know that $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu' \models_{\overline{\kappa}} \neg\alpha$, which implies $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}, \nu' \not\models_{\overline{\kappa}} \alpha$. So, by the inductive hypothesis we know that $\mathfrak{M}, \nu \not\models_{\overline{\kappa}_0} \alpha$. Now, because $\mathrm{Vocab}(\overline{\kappa}, \alpha) \subseteq \mathit{Vocab}(\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)})$ and because $\mathit{Vocab}(\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}) \subseteq \mathit{Vocab}(\mathfrak{M})$, the preconditions for the satisfaction relation are met. From this it follows that $\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\alpha$.

Case implication: The proof proceeds in the same way as for the case of negation.

Case `ist`: We need to show that if $\mathrm{Vocab}(\overline{\kappa}, \alpha) \subseteq \mathrm{Vocab}(\overline{\kappa}_0, \phi)$ then

$$\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \alpha) \quad \text{iff} \quad \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \alpha).$$

For the ($\Rightarrow$) direction we know that $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1, \alpha)$. Therefore, $\mathrm{Vocab}(\overline{\kappa}, \mathtt{ist}(\kappa_1, \alpha)) \subseteq \mathit{Vocab}(\mathfrak{M})$ and $\mathfrak{M} \models_{\overline{\kappa} * \kappa_1} \alpha$. By the inductive hypothesis $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa} * \kappa_1} \alpha$. Because $\mathrm{Vocab}(\overline{\kappa} * \kappa_1, \alpha) = \mathrm{Vocab}(\overline{\kappa}, \mathtt{ist}(\kappa_1, \alpha))$ and because $\mathit{Vocab}(\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)})$ contains all the contexts relevant to ϕ (and hence also to α), we know that the preconditions for the satisfaction relation are met. Hence $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1,\alpha)$.

For the other direction we know that $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1,\alpha)$, and hence we know that $\mathrm{Vocab}(\overline{\kappa},\mathtt{ist}(\kappa_1,\alpha)) \subseteq \mathit{Vocab}(\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)})$ and $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)} \models_{\overline{\kappa}*\kappa_1} \alpha$. By inductive hypothesis we know that $\mathfrak{M} \models_{\overline{\kappa}*\kappa_1} \alpha$. Because $\mathrm{Vocab}(\overline{\kappa}*\kappa_1,\alpha) = \mathrm{Vocab}(\overline{\kappa},\mathtt{ist}(\kappa_1,\alpha))$. And, because $\mathit{Vocab}(\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\phi)}) \subseteq \mathit{Vocab}(\mathfrak{M})$, we know that the preconditions for the satisfaction relation are satisfied. Hence $\mathfrak{M} \models_{\overline{\kappa}} \mathtt{ist}(\kappa_1,\alpha)$. $\square_{\textbf{finite-model-property}}$

Corollary (decidability): There is an effective procedure which will determine whether or not a formula given in some context is valid.

Proof (decidability): A formula, ϕ, given in $\overline{\kappa}_0$ is valid if and only if $\neg\phi$ given in $\overline{\kappa}_0$ is not satisfiable. So, to check if ϕ is valid it is sufficient to check if $\neg\phi$ is satisfiable. This is done by first generating all the models of the form $\mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\neg\phi)}$. There are finitely many models of this form. Each one of these models has only finitely many truth assignments, and every truth assignment is variable over finitely many propositional atoms (determined by the recursive function Vocab). So, all such models can be effectively generated in a finite amount of time. Once these models are generated we can determine the validity of the formula ϕ in the following way. If $\neg\phi$ is satisfied by any of the generated models then ϕ is not valid. On the other hand, if none of the generated models satisfy the formula $\neg\phi$, then ϕ is valid because we know that:

$$\forall \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\neg\phi)} \quad \mathfrak{M}_{\mathrm{Vocab}(\overline{\kappa}_0,\neg\phi)} \not\models_{\overline{\kappa}_0} \neg\phi,$$

which implies (by the finite model property):

$$\forall \mathfrak{M} \quad \mathfrak{M} \not\models_{\overline{\kappa}_0} \neg\phi.$$

So, ϕ is valid. $\square_{\textbf{decidability}}$

6 Comparison to Kripke Semantics

In this section we study the relationship between the semantics of context (as given in this paper) and Kripke semantics. We show that if all the aspects of partiality in the definition of a context model are disregarded, then most context models can be matched up to a particular class of Kripke models. Since there is always some leeway in the match, based on how we define the Kripke model and what notion of equivalence between context and Kripke models is taken, the models will not be matched precisely. We will discuss the adequacy of the match in more detail later.

We proceed to define some preliminaries for our construction.

Definition (non-partial model): A non-partial context model $\mathfrak{M}$ is a function which maps every context sequence $\overline{\kappa} \in \mathbb{K}^*$ to a set of total truth assignments,

$$\mathfrak{M} \in (\mathbb{K}^* \to \mathbf{P}(\mathbb{P} \to 2)).$$

Note that the two additional side conditions in the definition of the model are no longer needed. Also note that for the non-partial models the following property of the satisfaction relation holds:

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\phi \quad \text{iff} \qquad \text{not} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \phi.$$

Since in the comparison to Kripke semantics we will only be concerned with non-partial models, henceforth in this section we will refer to a partial context model simply as a context model. In this section we will also use the term "context logic" to refer to the general system described in §2 with the semantics restricted to non-partial models (since we are only doing model theory in this section, we are disregarding the formal system).

We now give a brief sketch of a standard propositional modal logic. We will be using a propositional modal logic with a countable number of modalities and its corresponding Kripke semantics. Given a context language specified by a possibly finite set of contexts, $\mathbb{K}$, and a set of propositional atoms, $\mathbb{P}$, we define a modal language consisting of the propositional atoms, $\mathbb{P}$, standard propositional connectives, $\neg$ and $\rightarrow$, and modalities, $\Box_1, \Box_2, \ldots$; one for each context from $\mathbb{K} = \{\kappa_\beta\}_{\beta<\alpha}$. We also define a bijective translation function which to each formula of the context logic, $\phi \in \mathbb{W}$, assigns a well-formed modal formula, $\phi^\Box$. The formula $\phi^\Box$ is obtained from ϕ by replacing each occurrence of $\mathtt{ist}(\kappa_\beta, \psi)$ in ϕ with $\Box_\beta(\psi^\Box)$. A *Kripke model* is a tuple $<\mathbb{S}, w_0, \pi, R_\beta>_{\beta<\alpha}$ where $\mathbb{S}$ is the set of *possible worlds*, $w_0 \in \mathbb{S}$ is the *actual world*, and π is a mapping from the the worlds in $\mathbb{S}$ to truth assignments over atomic propositions in $\mathbb{P}$, for some $\alpha \leq \omega$. Every R_β is a binary relation on $\mathbb{S}$. In order to distinguish Kripke models from context models, we use $\mathfrak{M}^\Box$ to refer to Kripke models. Kripke models are often called Kripke structures or possible-world structures. *Satisfaction* is defined to be a relation on a Kripke model, a world from that model, and a formula; it is written as $\mathfrak{M}^\Box, w \models \phi$. Note that the same symbol is used for satisfaction in the context logic, however it will be obvious from the arguments of the relation which satisfaction relation is being referred to. Atomic formulas are satisfied at a world if they are made true by the truth assignment associated with that world. Satisfaction for propositional connectives is defined as in classical propositional logic. The formula $\Box_\beta\,\phi$ is satisfied at a world w iff ϕ is satisfied at every world w' s.t. $w\,R_\beta\,w'$.

In order to compare a context model and a Kripke model, we need to know which worlds are intended to describe the same state of affairs as a given context, i.e. which worlds are associated to which context.

Definition (association relation): A relation A is an association relation from the context model $\mathfrak{M}$ to the Kripke model $\mathfrak{M}^\Box = <\mathbb{S}, w_0, \pi, R_\beta>_{\beta<\alpha}$ iff

1. $A \subseteq \mathbb{K}^* \times (\mathbb{P} \rightarrow 2) \times \mathbb{S}$
2. $(\forall\overline{\kappa})(\forall\nu \in \mathfrak{M}(\overline{\kappa}))(\exists w) \quad A(\overline{\kappa}, \nu, w)$
3. $A(\overline{\kappa}, \nu, w)$ implies $(\forall w')(\exists\beta)$ $(wR_\beta w'$ implies $(\exists\kappa')(\exists\nu')$ $A(\overline{\kappa} * \kappa', \nu', w'))$
4. $A(\overline{\kappa}, \nu, w)$ implies $\pi(w) = \nu$
5. $A(\epsilon, \pi(w_0), w_0)$

Given a context sequence $\overline{\kappa}$ and a truth assignment from $\mathfrak{M}(\overline{\kappa})$, the association relation expresses which world is to be associated with the truth assignment in that context, and vice versa. Note that the same truth assignment in different context sequences may produce different worlds.

In order to be able to compare a context model and a Kripke model, we need a notion of what it means for the two models to be equivalent.

Definition (elementary equivalence): A context model $\mathfrak{M}$ is elementarily equivalent to a Kripke model $\mathfrak{M}^\Box$ with respect to association A $(\mathfrak{M} \equiv \mathfrak{M}^\Box \text{ w.r.t. } A)$ iff

$$A(\overline{\kappa}, \nu, w) \quad \text{implies} \quad (\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi \quad \text{iff} \quad \mathfrak{M}^\Box, w \models \phi^\Box)$$

for any context sequence $\overline{\kappa}$, truth assignment $\nu \in \mathfrak{M}(\overline{\kappa})$, world $w \in \mathbb{S}$, and wff ϕ.

We define some properties of relations which will be used to define classes of Kripke models.

Definition (properties of relations):

$\Phi_1(R_i, R_j) : (\forall w_1 \forall w_2 \forall w_2' \forall w_3) \quad ((w_1 R_i w_2 \; \& \; w_2 R_j w_3 \; \& \; w_1 R_i w_2') \Rightarrow w_2' R_j w_3)$

$\Phi_2(R) : \quad (\forall w)(\exists w') \quad wRw'$ **(seriality)**

$\Phi_3(R) : \quad (\forall w_1 \forall w_2 \forall w_2')(w_1 R w_2 \; \& \; w_1 R w_2') \Rightarrow w_2 = w_2'$ **(functionality)**

$\Phi_4(R_i, R_j) : \Phi_2(R_i) \; \& \; \Phi_2(R_j) \; \& \; (\forall w_1 \forall w_2 \forall w_2')(w_1 R_i w_2 \Rightarrow (w_2 R_j w_2' \Leftrightarrow w_1 R_j w_2'))$

Let $\mathsf{C}_1, \ldots, \mathsf{C}_4$ be the classes of Kripke models in which all the accessibility relations (or all the ordered pairs of accessibility relations) satisfy conditions $\Phi_1, \ldots, \Phi_4$ respectively. Note that in the standard modal correspondence theory the relation Φ_1 corresponds to the axiom schema $\triangle$.

We also introduce a class of context models, called the *actual models.* They have the property that the empty context sequence is associated with a single truth assignment, which is interpreted as the actual state of affairs or the state of affairs in the actual world.

Definition ($\mathfrak{Actual}$): A context model $\mathfrak{M}$ is an *actual model*, $\mathfrak{M} \in \mathfrak{Actual}$, iff $|\mathfrak{M}(\epsilon)| = 1$.

6.1 Representing Kripke Models with Context Models

Theorem (representation1): For every context model $\mathfrak{M} \in \mathfrak{Actual}$, there exists a Kripke model $\mathfrak{M}^\Box$, and an association A from $\mathfrak{M}$ to $\mathfrak{M}^\Box$ such that $\mathfrak{M} \equiv \mathfrak{M}^\Box$ w.r.t. A.

Proof (representation1): Given a context model $\mathfrak{M}$, we construct an elementarily equivalent Kripke model $\mathfrak{M}^\Box$. Intuitively, for every truth assignment in every context sequence, we create a world in the Kripke model. The function π of a world is the same as the truth assignment that created that world. We then define relations on these worlds and show that the created structure, $\mathfrak{M}^\Box$, is in fact a Kripke model. Finally, we prove that the two models are in fact elementarily equivalent.

The construction of $\mathfrak{M}^\Box$ proceeds in four stages. Firstly, we define a set of objects $\mathbb{S}$, one associated with every ν from every context sequence $\overline{\kappa}$. These objects will constitute the set of worlds of what will be the Kripke model $\mathfrak{M}^\Box$; formally

$$\mathbb{S} := \{<\overline{\kappa}, \nu> \,|\, \overline{\kappa} \in \mathbb{K}^*,\ \nu \in \mathfrak{M}(\overline{\kappa})\}.$$

Similarly, an association A from $\mathfrak{M}$ to $\mathfrak{M}^\Box$ is defined

$$A := \{<\overline{\kappa}, \nu, <\overline{\kappa}, \nu>> \,|\, \overline{\kappa} \in \mathbb{K}^*,\ \nu \in \mathfrak{M}(\overline{\kappa})\}.$$

Secondly, we associate a truth assignment with every element of $\mathbb{S}$. We define the truth assignment associated with the world $<\overline{\kappa}, \nu> \in \mathbb{S}$ to be ν. Thirdly, the accessibility relation is defined to capture the structure of the groupings of truth assignments into contexts, and the relations among contexts. A world created by some context sequence $\overline{\kappa}$ will be in relation R_i to all the worlds created by the context sequence $\overline{\kappa} * \kappa_i$. Formally,

$$wR_iw' \quad \text{iff} \quad \exists\overline{\kappa}\exists\nu\exists\nu'\,(w = <\overline{\kappa}, \nu> \quad \text{and} \quad w' = <\overline{\kappa} * \kappa_i, \nu'>).$$

Finally, in the fourth stage, we define the actual world to be $<\epsilon, \nu> \in \mathbb{S}$, the world associated with the empty sequence. Note that this world is unique, since $\mathfrak{M} \in \mathfrak{Actual}$. Putting together all of the above components we define the Kripke model $\mathfrak{M}^\Box := <\mathbb{S}, w_0, \pi, R_\beta>_{\beta<|\mathbb{K}|}$. It is clear that $\mathfrak{M}^\Box$ is in fact a Kripke model.

Furthermore, note following properties of $\mathfrak{M}^\Box$:

1. The world associated with ϵ, the root of the tree (defined by the domain of the context model $\mathfrak{M}$), has no predecessors:

   $$w = <\epsilon, \nu> \in \mathbb{S} \quad \text{implies} \quad \forall\, w' \in \mathbb{S} \quad \neg(w'\; R_i\; w)$$

   for every relation R_i in the Kripke model $\mathfrak{M}^\Box$.

2. If $\overline{\kappa} * \kappa_i$ is an inconsistent context then the worlds associated with $\overline{\kappa}$ will not be related to any worlds via R_i. Formally, if $\mathfrak{M}(\overline{\kappa} * \kappa_i) = \emptyset$ then

   $$w = <\overline{\kappa}, \nu> \in \mathbb{S} \quad \text{implies} \quad \forall\, w' \in \mathbb{S} \quad \neg(w\; R_i\; w').$$

All that remains to be shown is that $\mathfrak{M} \equiv \mathfrak{M}^\Box$ w.r.t. A, i.e. that

$$A(\overline{\kappa}, \nu, w) \quad \text{implies} \quad (\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi \quad \text{iff} \quad \mathfrak{M}^\Box, w \models \phi^\Box).$$

This is proved by induction on the complexity of the formula ϕ.

Case atomic: $\phi = \rho$. Since the truth assignment $\pi(w)$ is the same as ν when $A(\overline{\kappa}, \nu, w)$, and since $\rho^\Box$ is the same atom as ρ, it clearly follows that $\mathfrak{M}, \nu \models_{\overline{\kappa}} \rho$ iff $\mathfrak{M}^\Box, w \models \rho^\Box$

Case negation: $\phi = \neg\psi$. We begin by assuming $A(\overline{\kappa}, \nu, w)$ and

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \neg\psi.$$

By definition of satisfaction for the context logic, this holds iff

$$\text{not} \quad (\mathfrak{M}, \nu \models_{\overline{\kappa}} \psi).$$

Which by inductive hypothesis holds iff

$$\text{not} \quad (\mathfrak{M}^{\Box}, w \models \psi^{\Box}),$$

and by the satisfaction relation for modal logic, this is true iff

$$\mathfrak{M}^{\Box}, w \models \neg\psi^{\Box}.$$

Finally, since negation translates into negation, the above is true iff

$$\mathfrak{M}^{\Box}, w \models (\neg\psi)^{\Box}.$$

Case implication: $\phi = \psi \to \chi$. We begin by assuming $A(\overline{\kappa}, \nu, w)$ and

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \psi \to \chi.$$

By definition of satisfaction for the context logic this holds iff

$$\mathfrak{M}, \nu \models_{\overline{\kappa}} \psi \quad \text{implies} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \chi,$$

which by inductive hypothesis holds iff

$$\mathfrak{M}^{\Box}, w \models \psi^{\Box} \quad \text{implies} \quad \mathfrak{M}^{\Box}, w \models \chi^{\Box}.$$

By the satisfaction relation for modal logic, this is true iff

$$\mathfrak{M}^{\Box}, w \models \psi^{\Box} \to \chi^{\Box}.$$

Finally, since implication translates into implication, the above is true iff

$$\mathfrak{M}^{\Box}, w \models (\psi \to \chi)^{\Box}.$$

Case `ist` $\Rightarrow$: We begin by assuming $A(\overline{\kappa}, \nu, w)$ and

$$\text{not} \quad \mathfrak{M}^{\Box}, w \models (\mathtt{ist}(\kappa_i, \psi))^{\Box}.$$

By definition of translation function, it is the case that

$$\text{not} \quad \mathfrak{M}^{\Box}, w \models \Box_i \psi^{\Box}.$$

Therefore, there exists w' such that wR_iw' and

$$\text{not} \quad \mathfrak{M}^{\Box}, w' \models \psi^{\Box}.$$

Furthermore, there exists κ_i and there exists $\nu' \in \mathfrak{M}(\overline{\kappa} * \kappa_i)$ such that $w' = \langle\overline{\kappa} * \kappa_i, \nu'\rangle$ and $A(\overline{\kappa} * \kappa_i, \nu', w')$. By inductive hypothesis,

$$\text{not} \quad \mathfrak{M}, \nu' \models_{\overline{\kappa} * \kappa_i} \psi.$$

Therefore, by definition of satisfaction,

$$\text{not} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \mathtt{ist}(\kappa_i, \psi).$$

Case ist ⇐: We begin by assuming $A(\overline{\kappa}, \nu, w)$ and

$$\text{not} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \mathtt{ist}(\kappa_i, \psi).$$

Therefore, by definition of satisfaction for the context logic, for some $\nu' \in \mathfrak{M}(\overline{\kappa} * \kappa_i)$

$$\text{not} \quad \mathfrak{M}, \nu' \models_{\overline{\kappa} * \kappa_i} \psi.$$

Now put $w' = <\overline{\kappa} * \kappa_i, \nu'>$. Therefore, by construction, $A(\overline{\kappa} * \kappa_i, \nu', w')$ and $wR_i w'$. By inductive hypothesis,

$$\text{not} \quad \mathfrak{M}^{\Box}, w' \models \psi^{\Box}.$$

Now, by definition of satisfaction for modal logic,

$$\text{not} \quad \mathfrak{M}^{\Box}, w \models \Box_i \psi^{\Box}.$$

By the definition of the translation function this holds iff

$$\text{not} \quad \mathfrak{M}^{\Box}, w \models (\mathtt{ist}(\kappa_i, \psi))^{\Box}.$$

□**representation1**

6.2 Representing Context Models with Kripke Models

We will not be able to represent all Kripke models using context models. Thus we identify the class of Kripke models which can be represented using context models. This turns out to be C_1, which is also the class of Kripke models which satisfy the $\triangle$ axiom schema.

Theorem (representation2): For every Kripke model $\mathfrak{M}^{\Box} \in \mathsf{C}_1$ there exists a context model $\mathfrak{M}$ such that $\mathfrak{M} \equiv \mathfrak{M}^{\Box}$ w.r.t. A.

Proof (representation2): Given a Kripke model $\mathfrak{M}^{\Box}$ we construct an elementarily equivalent context model $\mathfrak{M}$. Intuitively, we will identify context sequences with paths through the Kripke model $\mathfrak{M}^{\Box}$. Then, the truth assignments of the worlds in $\mathfrak{M}^{\Box}$ which can be reached via a path $\overline{\kappa}$ will be placed in $\mathfrak{M}(\overline{\kappa})$.

Assume $\mathfrak{M}^{\Box} = <\mathbb{S}, w_0, \pi, R_\beta>_{\beta<\alpha}$ is the Kripke model we are transforming. First we define the set of contexts $\mathbb{K}$. This set is identified with the set of all the relations from the Kripke model

$$\mathbb{K} := \{R_\beta\}_{\beta<\alpha}.$$

Now we define the association relation A. The empty sequence ϵ will be associated with the actual world w_0. For other context sequences, the association is defined inductively. If w_i is some world associated with $\overline{\kappa}$, then all the worlds w' related to w via R_i are associated with $\overline{\kappa}*R_i$. For example, if in $\mathfrak{M}^\Box$ some world w_5 is associated with $[w_2, R_8, R_5, R_1]$, and $w_5R_4w_9$ in the Kripke model, then the world w_9 will be associated with $[w_2, R_8, R_5, R_1, R_4]$. Thus the context sequence $[w_2, R_8, R_5, R_1, R_4]$ is associated with all the worlds which can be reached in $\mathfrak{M}^\Box$ by starting at w_2 and then making an R_8 transition (in the Kripke model), followed by an R_5 transition, followed by R_1 and R_4 transitions. Formally:

$$A(\epsilon, \pi(w_0), w_0)$$

$$A(\overline{\kappa} * R_i, \pi(w), w) \quad \text{iff} \quad \exists w' \in \mathbb{S} \quad A(\overline{\kappa}, \pi(w'), w') \quad \text{and} \quad w'R_iw.$$

It is not difficult to verify that the relation A, as defined above, is indeed an association relation. Once we have the association relation, the context model is defined in the obvious way:

$$\mathfrak{M}(\overline{\kappa}) = \{\pi(w) \,|\, A(\overline{\kappa}, \pi(w), w)\}.$$

Now we state a property of the construction.

Lemma (association):

$$\forall w\ \forall w'\ (A(\overline{\kappa}, \pi(w), w) \quad \text{and} \quad A(\overline{\kappa} * R_i, \pi(w'), w')) \quad \text{implies} \qquad wR_iw'.$$

The lemma expresses that all the worlds associated with $\overline{\kappa}$ are related via R_i (in $\mathfrak{M}^\Box$) to all the worlds associated with $\overline{\kappa} * R_i$. The lemma is a consequence of (1) the construction of A, in which a world is added to $\overline{\kappa} * R_i$ if it is connected via R_i to *some* world from $\overline{\kappa}$, and (2) the fact that the Kripke model $\mathfrak{M}^\Box \in \mathsf{C}_1$, i.e. satisfies the relation Φ_1. We postpone the proof till later.

All that remains to be shown is that $\mathfrak{M} \equiv \mathfrak{M}^\Box$ w.r.t. A. We do this by proving the stronger claim

$$A(\overline{\kappa}, \nu, w) \quad \text{implies} \quad (\mathfrak{M}, \nu \models_{\overline{\kappa}} \phi \quad \text{iff} \quad \mathfrak{M}^\Box, w \models \phi^\Box).$$

The proof is by induction on the structure of the formula ϕ. We skip all the easy cases and we show the `ist` case.

Case `ist` $\Rightarrow$: We begin by assuming

$$A(\overline{\kappa}, \nu, w) \quad \text{and} \qquad \text{not} \quad \mathfrak{M}^\Box, w \models \Box_i\psi^\Box.$$

Therefore, by definition of satisfaction for modal logic, $\exists w' \in \mathbb{S}$ s.t.

$$wR_iw' \quad \text{and} \qquad \text{not} \quad \mathfrak{M}^\Box, w \models \psi^\Box.$$

Now $A(\overline{\kappa} * R_i, \pi(w'), w')$ by construction of A. So by inductive hypothesis

$$\text{not} \quad \mathfrak{M}, \pi(w') \models_{\overline{\kappa}*R_i} \psi.$$

Therefore, by definition of satisfaction for context logic

$$\text{not} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \mathtt{ist}(R_i, \psi).$$

Case ist $\Leftarrow$: We begin by assuming

$$A(\overline{\kappa}, \nu, w) \quad \text{and} \qquad \text{not} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \mathtt{ist}(R_i, \psi).$$

Thus by definition of satisfaction for the context logic, $\exists \nu' \in \mathfrak{M}(\overline{\kappa} * R_i)$ s.t.

$$\text{not} \quad \mathfrak{M}, \nu' \models_{\overline{\kappa}*R_i} \psi.$$

Now $\mathfrak{M}(\overline{\kappa} * R_i) = \{\pi(w) \mid A(\overline{\kappa} * R_i, \pi(w'), w')\}$. Thus $\nu' = \pi(w')$ and $A(\overline{\kappa} * R_i, \nu', w')$ for some $w' \in \mathbb{S}$. Now by (**association lemma**) wR_iw' and by inductive hypothesis

$$\text{not} \quad \mathfrak{M}^\Box, w' \models \psi^\Box.$$

Therefore, by satisfaction relation for modal logic,

$$\text{not} \quad \mathfrak{M}^\Box, w \models \Box_i\psi^\Box.$$

$\Box_{\textbf{representation2}}$

Now we prove the (**association lemma**).

Proof (association lemma): The proof is by induction on the length of the sequence $\overline{\kappa}$.

Case base: $\overline{\kappa}$ is the empty sequence, ϵ. Then the actual world, w_0, is the only world associated with $\overline{\kappa}$. The representation function is defined so that all the worlds related to w_0 via R_i will be associated to $\overline{\kappa} * R_i$. Since w_0 is the only world associated with $\overline{\kappa}$ then it trivially follows that all the worlds associated with $\overline{\kappa}$ (namely w_0) are related via R_i to all the worlds associated to $\overline{\kappa} * R_i$.

Case inductive step: Assume $A(\overline{\kappa}_2, \pi(w_2), w_2)$ and $A(\overline{\kappa}_3, \pi(w_3), w_3)$. Since $\overline{\kappa}_2 < \epsilon$, we let $\overline{\kappa}_2 := \overline{\kappa}_1 * R_i$. Now $A(\overline{\kappa}_2, \pi(w_2), w_2)$ implies that there exists $w_1 \in S$ such that $A(\overline{\kappa}_1, \pi(w_1), w_1)$ and $w_1R_iw_2$. By induction hypothesis, it follows that for any $w_2' \in S$ such that $A(\overline{\kappa}_2, \pi(w_2'), w_2')$, $w_1R_iw_2'$. By definition of A it follows that for some $w_2' \in S$ such that $A(\overline{\kappa}_2, \pi(w_2'), w_2')$, $w_2'R_jw_3$. Now by the restriction on the accessibility relation dictated by the fact that $\mathfrak{M}^\Box \in \mathsf{C}_1$, we get that $w_2R_jw_3$. $\Box_{\textbf{association}}$

6.3 Discussion

The representation results in this section depend on our definition of the Kripke model and the definition of elementary equivalence. Variations in either of these definitions would slightly change the theorems. For example, sometimes the actual world is excluded from the definition of the Kripke model. In this case we could generalize the (**representation1 theorem**) to hold for any context model, rather than only those in $\mathfrak{Actual}$. For (**representation2 theorem**) to hold we would need a way of including multiple subtrees in a single context model. One solution would be to connect all the subtrees to ϵ and associate the empty vocabulary with the empty sequence. To take another example, stronger results could be obtained by insisting that the association relation matches every world to some truth assignment in some context sequence. But here again, to prove (**representation2 theorem**) we would need vocabularies or a stronger notion of a context model which would allow multiple rooted subtrees as the domain of $\mathfrak{M}$. To conclude, there is always some leeway in representation results, based on the basic definitions. The main purpose of this section is to give the general flavor of the relations in expressiveness of the two kinds of models and a methodology which can be used for comparing context models and Kripke models.

6.4 Correspondence Results

In this subsection we use the previously established general relations between context models and Kripke models to relate some interesting classes of context models to classes of Kripke models. Essentially, this amounts to matching some intuitive notions of context to accessibility relations between worlds.

Proposition (correspondences): If $\mathfrak{M} \equiv \mathfrak{M}^{\Box}$ w.r.t. A, then the following hold

1. $\mathfrak{M}^{\Box} \in \mathsf{C}_2$ implies $\mathfrak{M} \in \mathfrak{Consistent}$
2. $\mathfrak{M}^{\Box} \in \mathsf{C}_3$ implies $\mathfrak{M} \in \mathfrak{Truth}$
3. $\mathfrak{M}^{\Box} \in \mathsf{C}_4$ implies $\mathfrak{M} \in \mathfrak{Flat}$

Proof (correspondence (1)): Assume $\mathfrak{M} \notin \mathfrak{Consistent}$. Since $\mathfrak{M} \equiv \mathfrak{M}^{\Box}$ w.r.t. A, there exists w s.t. $A(\epsilon, \pi(w), w)$. Thus $\mathfrak{M}(\epsilon) \neq \emptyset$. Since $\mathfrak{M} \notin \mathfrak{Consistent}$, there exists $\overline{\kappa} * \kappa_i$ s.t. $\mathfrak{M}(\overline{\kappa} * \kappa_i) = \emptyset$. Choose such a $\overline{\kappa} * \kappa_i$ so that $\mathfrak{M}(\overline{\kappa}') \neq \emptyset$ for $\overline{\kappa} * \kappa_i < \overline{\kappa}' \leq \epsilon$. Therefore,

$$\text{not} \quad \mathfrak{M}, \nu \models_{\overline{\kappa}} \neg \mathtt{ist}(\kappa_i, \bot).$$

Now since $\mathfrak{M} \equiv \mathfrak{M}^{\Box}$ w.r.t. A, there exists $w \in \mathbb{S}$ s.t. $A(\overline{\kappa}, \nu, w)$ and

$$\text{not} \quad \mathfrak{M}^{\Box}, w \models (\neg \mathtt{ist}(\kappa_i, \bot))^{\Box}.$$

Therefore, by definition of language translation we get

$$\text{not} \quad \mathfrak{M}^{\Box}, w \models \neg \Box_i \bot.$$

Therefore, as we know from modal logic, $\mathfrak{M}^{\Box} \notin \mathsf{C}_2$, a contradiction. $\Box_{\mathbf{correspondence(1)}}$

The proofs of the remaining two propositions are identical, with the exception that a different formula is used to reach a contradiction. In case 2, we use the fact that a Kripke model $\mathfrak{M}^{\Box}$ s.t.

$$\text{not} \quad \mathfrak{M}^{\Box}, w \models \Box_i\phi \vee \neg\Box_i\phi$$

is not in C_3. This can easily be proved using standard methods for deriving correspondence results in modal logic. The formula for the third proposition is somewhat more complex:

$$\text{not} \quad \mathfrak{M}^{\Box}, w \models (\Box_i\Box_j\phi \rightarrow \Box_j\phi) \wedge (\Box_i\neg\Box_j\phi \rightarrow \neg\Box_j\phi)$$

which corresponds to axiom schemas (4^{-1}) and (5^{-1}). That the above model $\mathfrak{M}^{\Box}$ is not in C_4 can again be showed using standard methods for correspondence results in modal logic or using some equivalences we have proved earlier. As we know from §3.3, any context system containing axioms schemas (4^{-1}) and (5^{-1}) is equivalent to a system containing schemas (4), (5), and (**D**). Since the correspondence results for these schemas are well known in modal logic, it is again simple to verify that they exactly characterize the class of Kripke models C_4.

7 Related Work

Our work is largely based on McCarthy's ideas on context. McCarthy's research [16, 18] in formalizing common sense has led him to believe that in order to achieve human-like generality in reasoning we need to develop a formal theory of context. The key idea in McCarthy's proposal was to treat contexts as formal objects, enabling us to state that a proposition is true in a context: $\mathtt{ist}(\kappa, \phi)$ where ϕ is a proposition and κ is a context. This permits axiomatizations in a limited context to be expanded so as to *transcend* their original limitations.

There has been other research done in this area; most notable is the work of Lifschitz, Shoham, Guha, Giunchiglia, and Attardi and Simi. We briefly treat each in turn.

Two contexts can differ in, at least, three ways: they may have different vocabularies; or they may have the same vocabulary but describe different states of affairs, or (in the first order case) they may have the same vocabulary (i.e. language) but treat it differently (i.e the arities may not be the same). The first two differences were studied in [3], and led to two different views on the use of context. Lifschitz's early note on formalizing context [15] concentrates on the third difference. Shoham, in his work on contexts, concentrates on the second difference [19]. Every proposition is meaningful in every context, but the same proposition can have different truth values in different contexts. Shoham approached the task of formalizing context from the perspective of modal and non-classical logics. He defines a propositional language with an analogue to the $\mathtt{ist}$ modality, and a relation $\kappa_1 \bullet\!\!\supset \kappa_2$, expressing that context κ_1 is as general as context κ_2. Drawing on the intuitive analogy between a context κ and the proposition *current-context-is*(κ), Shoham identifies the set of contexts with the set of propositions. This enables him to define truth in a context $\mathtt{ist}(\kappa, p)$, in terms of the the conditional *current-context-is*$(\kappa) \rightarrow p$, where $\rightarrow$ is interpreted as as some form of intuitionistic or relevance implication. His paper gives a list of 14 benchmark sentences which characterize this implication.

Guha's dissertation contains a number of examples of context use. These demonstrate how reasoning with contexts should behave, and which properties a formalization of context should exhibit. The Cyc knowledge base [13], which is the main motivation for Guha's context research, is made up of many theories, called *micro-theories*, describing different aspects of the world. Guha has tailored the design of micro-theories after contexts. The examples of context use given in his dissertation are especially interesting because they: 1) are motivated by a real system, and portray situations that arise in practice. 2) are implemented and work in Cyc.

Following up on Weyhrauch's ideas [20], Giunchiglia proposes a context framework called *multilanguage* systems [10]. Although different contexts are allowed different vocabularies, Giunchiglia's framework is significantly weaker than the systems described above because his language does not include an `ist` modality. Instead he introduces *bridge rules*, special kind of inference rules which allow formulas in one context to be inferred based on facts derived in another context. Giunchiglia's research emphasizes a proof theoretic approach. Although never made precise, his semantics seems to have a strong procedural flavor.

Attardi and Simi's motivation for the theory of *viewpoints*, [1], is similar to those of contexts. However, their formalization differs from ours in a number of ways.

They have a syntactic approach to modality. This means that rather then introducing a modal operator, they extend the ontology by introducing names of sentences as first-class objects. Thus their equivalent of the `ist` modality becomes a regular predicate. The difficulty with this approach is that it allows the possibility of self-reference and thus opens the doors to paradoxes. A significant portion of Attardi and Simi's work is focused at avoiding paradoxes. They accomplish this by weakening the rule corresponding to the context switching rule (**CS**). Thus the fact that $\mathtt{ist}(\kappa_1, p)$ holds in some context sequence $\overline{\kappa}$ does not always imply that p will hold in the context sequence $\overline{\kappa} * \kappa_1$.

The main divergence from McCarthy's, Guha's and our notions of context, is that their viewpoint is not a primitive formal object, but a set of names of sentences. Intuitively, this set corresponds to the assumptions made in that viewpoint. Consequently the logic of viewpoints contains axioms like $\mathtt{ist}(\kappa, p) \rightarrow (\kappa \rightarrow p)$. This approach is similar to formalization proposed by Shoham.

Also, they assume that all viewpoints have the same language, thus avoiding partiality in their logic.

7.1 Comparison to Logic of Belief

There is also a clear parallel between the logic of context and the modal logic of belief [14]. The modality $\mathtt{ist}(\kappa, \phi)$ may be interpreted as expressing that the agent κ knows or believes the sentence ϕ. However, there are fundamental semantic distinctions between the two logics. The `ist` modality is meant to capture what is actually true or valid in a context rather than what is believed to be true in a context. This is manifested by the ($\triangle$) axiom schema, which can be written as

$$\mathtt{ist}(\kappa_1, \mathtt{ist}(\kappa_2, \phi)) \vee \mathtt{ist}(\kappa_1, \neg\mathtt{ist}(\kappa_2, \phi))$$

if we disregard the vocabulary restrictions. Intuitively, it tells us that a context is committed on what is valid in another context. The corresponding schema is not true of belief. There is no reason why one agent should be committed on the beliefs of other agents. The justification for our view might best be understood when thinking about contexts as knowledge bases. Then $\mathtt{ist}(\kappa, \phi)$ holds iff the formula ϕ is actually valid in the κ knowledge base. Thus the ($\triangle$) schema expresses that it is true in κ_1 knowledge base that ϕ is valid in the the κ_2 knowledge base or that it is true in the κ_1 knowledge base that ϕ is not valid in the κ_2 knowledge base. In other words, the κ_1 knowledge base behaves as if though it can see into the κ_2 knowledge base and thus decide for any formula ϕ whether or not it is valid in κ_2.

Another distinction between the logic of context and the logic of belief is in the formal semantics commonly ascribed them. Logics of belief is commonly ascribed Kripke semantics. Modeling truth or validity in a context by a Kripke model, i.e. by relation between worlds would not be intuitive because we want contexts to be reified as first class objects in the semantics. This will allow us (in the predicate case) to state relations between contexts, define operations on contexts, and specify how sentences from one context can be *lifted* into another context.

Finally, vocabulary considerations are not commonly addressed in logics of belief.

8 Conclusions and Future Work

Our goal is to extend the system to a full quantification logic. One advantage of quantificational system is that it enables us to express relations between context, operations on contexts, and state *lifting rules* which describe how a fact from one context can be used in another context. However, in the presence of context variables it might not be possible to define the vocabulary of a sentence without knowing which object a variable is bound to. Therefore the first step in this direction is to to examine propositional systems with dynamic definitions of meaningfulness.

We also plan to define non-Hilbert style formal systems for context. Probably the most relevant is a natural deduction system, which would be in line with McCarthy's original proposal of treating contextual reasoning as a strong version of natural deduction. In such a system, entering a context would correspond to making an assumption in natural deduction, while exiting a context corresponds to discharging an assumption.

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